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NOTES AND QUESTIONS.

Notes and Questions.

Note on the representation of a Number as the sum of two others satisfying given conditions.

1. The object of this note is to discuss some of the properties of the triads of numbers of the types (written in the usual arithmetical notation)

$$a_1\, a_2\, a_3 \cdots\cdots \cdots a_p,$$
$$b_1\, b_2\, b_3 \cdots\cdots\cdots b_p,$$
$$c_1\, c_2\, c_3 \cdots\cdots\cdots c_p,$$

in γ-ary scale $[\gamma = (3p + 1)]$, where no two of the a's, b's, c's are equal and none of them is zero and the sum of the first two numbers is equal to the third number.

It is, at once, obvious that all the numbers from 1 to $3p$ are distributed among the digits of the three numbers and that $c_1 \not< 3$.

Evidently $\Sigma a_\gamma + \Sigma b_\gamma + \Sigma c_\gamma = 1 + 2 + 3 + \cdots\cdots + (\gamma - 1)$

$$= \frac{\gamma(\gamma - 1)}{2}; \qquad \cdots \qquad \cdots \quad (1)$$

and $\gamma^{p-1}(a_1 + b_1 - c_1) + \gamma^{p-2}(a_2 + b_2 - c_2) + \cdots\cdots$

$$+ \gamma^{p-s}(a_s + b_s - c_s) + \cdots\cdots (a_p + b_p - c_p) = 0.$$

Now the successive values of $a_s + b_s - c_s$, $a_{s-1} + b_{s-1} - c_{s-1}$, ... admit of one or more of the following types of sequences :—

(i) zeroes

(ii) $\gamma, -1$

(iii) $\gamma, \gamma - 1, -1$

(iv) $\gamma, \gamma - 1, -\gamma - 1,$

etc.

Suppose the p numbers $a_1 + b_1 - c_1$, $a_2 + b_2 - c_2$, ... $a_p + b_p - c_p$ contain x zeroes, y_0 groups of the type (ii), y_1 groups of the type (iii), y_2 groups of the type (iv), and so on.

Then $\quad \Sigma a_\gamma + \Sigma b_\gamma - \Sigma c_\gamma = (\gamma - 1)(y_0 + 2y_1 + 3y_2 + \cdots\cdots)$

$$= (\gamma - 1)k, \quad \text{(say)};$$

$$i.e., \qquad y_0 + 2y_1 + 3y_2 + \ldots\ldots = k = \frac{\Sigma a_\gamma + \Sigma b_\gamma - \Sigma c_\gamma}{\gamma - 1} \ldots \quad (2)$$

Also, $\quad x + 2\,y_0 + 3\,y_1 + 4\,y_2 + \ldots\ldots = p. \ldots \qquad \ldots \quad (3)$

From (2) and (3),

$$x + y_0 + y_1 + y_2 + \ldots\ldots = p - k, \qquad \ldots \quad (4)$$

which shows that k may take all values from 0 to $p-1$.

From (1) and (2),

$$\Sigma c_\gamma = \frac{(\gamma-1)(\gamma-2k)}{4}, \qquad \ldots \qquad \ldots \quad (5)$$

which is one of the necessary conditions that $[c_1\, c_2 \ldots\ldots c_p]$ may admit of the representation in question.

The upper and lower limits of Σc_γ are obtained by putting $k = 0$ and $(p-1)$ respectively in (5);

$$i.e., \qquad \frac{(\gamma-1)(\gamma+8)}{12} < \Sigma c_\gamma < \frac{\gamma(\gamma-1)}{4}. \qquad \ldots \quad (6)$$

Hence,

(i) when $\gamma = 12\,m + 1$, k can take all integral values from 0 to $4\,m - 1$,

(ii) $\gamma = 12\,m + 4$, k must be even, for otherwise $(\gamma-1)(\gamma-2k)$ will not be divisible by 4; and hence k can take all even values from 0 to $4\,m$;

(iii) when $\gamma = 12\,m + 10$, k must be odd, for the same reason as before and therefore can take all odd values from 1 to $4m+1$;

(iv) when $\gamma = 12\,m + 7$, $(\gamma-1)(\gamma-2k)$ is not divisible by 4 and hence the representation in question is impossible.

2. When the number $(c_1\, c_2 \ldots\ldots c_p)$ admits of the representation in question,

$$\frac{(p-k)!}{x!\; y_0!\; y_1! \ldots\ldots}$$

permutations of the number also admit of a similar representation. For, we may so permute the digits $c_1, c_2, \cdots c_p$ that the numbers of each of the groups of digits corresponding to the types (i), (ii), (iii) ... of § 1, occur together though the relative positions of the different groups may be altered.

3. It is interesting to consider the particular case $\gamma = 10$ in some detail.

Equations (2), (3) and (5) above reduce in this case to

$$y_0 + 2y_1 = k\,;\ x + 2y_0 + 3y_1 = 3\,;\ \text{and } c_1 + c_2 + c_3 = \frac{9(5-k)}{2}.$$

Consequently, k cannot take the values 0 and 2. Since $k<3$, $k=1$.

Hence the only possible case is when

$$y_0 = 1,\ y_1 = 0,\ x = 1\,;$$

and

$$c_1 + c_2 + c_3 = 18.$$

Thus if $a_1\ a_2\ a_3 + b_1\ b_2\ b_3 = c_1\ c_2\ c_3$, the digits should satisfy either of the following sets of equations:—

$$\text{(I)}\quad \left.\begin{aligned} c_1 + c_2 + c_3 &= 18 \\ a_1 + b_1 - c_1 &= -1 \\ a_2 + b_2 - c_2 &= 10 \\ a_3 + b_3 - c_3 &= 0. \end{aligned}\right\} \qquad \text{(II)}\quad \left.\begin{aligned} c_1 + c_2 + c_3 &= 18 \\ a_1 + b_1 - c_1 &= 0 \\ a_2 + b_2 - c_2 &= -1 \\ a_3 + b_3 - c_3 &= 10. \end{aligned}\right\}$$

The four equations in (I) or (II) are not all independent; for, any one of them can be deduced from the other three. Thus it is sufficient for us to consider three equations only in (I) and (II).

We shall denote that representation of $c_1\,c_2\,c_3$ in which $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3$ by the term *primitive* representation. From each such primitive representation, we can get three more by permuting between a_1, b_1; a_2, b_2; and a_3, b_3.

It can be shown by examining the behaviour of the 2-partitions of $(c_1 - 1)$ and c_3 or of c_1 and $(c_2 - 1)$ that for given values of $c_1\,c_2\,c_3$ satisfying the condition $c_1 + c_2 + c_3 = 18$, there are at most two primary solutions of (I) or (II), while

neither may have a solution, as when $c_1 = 7$, $c_2 = 6$, $c_3 = 5$;
or (I) alone may have no solution, as when $c_1 = 9$, $c_2 = 8$, $c_3 = 1$;
or (II) .... $c_1 = 9$, $c_2 = 3$, $c_3 = 6$.

In no case does the total number of primitive representations of any number of the form $(c_1c_2c_3)$, viz., $[100c_1 + 10c_2 + 18 - (c_1 + c_2)]$ exceed two.

The number of solutions of (I) and (II) for all numbers of three digits whose sum is 18 satisfying our conditions is given below in a tabular form:—

Numbers.	No. of primitive solutions of (I)	No. of primitive solutions of (II)
918; 837; 819; 783; 675.	2	0
945; 864.	1	1
936; 927; 738; 729; 648; 639; 576; 549; 468; 459.	1	0
486; 972; 963; 954; 873; 495; 846; 792; 693; 657; 594.	0	1
981; 891; 567.	0	2
765; 756; 684; 396; 387; 378; 369; 297; 279; 198; 189.	0	0

NOTE.—As an exercise on the application of partitions of the type discussed here, one may investigate the arrangement of the digits 0, 1, 2, 3, ... 9 in such a way as to form a number divisible by all the numbers from 1 to 17. It is found that there are only *four* such arrangements, *viz*:

(i) 2438195760
(ii) 3785942160
(iii) 4876391520
(iv) 4753869120.

A. A. KRISHNASWAMI AYYANGAR.

A Note.

It may be of interest to note that the value of 2^{100} is

1267650600228229401496703205376.

Persons interested may attempt at factorising $(2^{100} + 1)$.

HANS R. GUPTA.

Solutions.

Question 1229.

(Prof. K. J. SANJANA):—Show how to find the equation of the surface traced out by the shortest distances of all points of a given straight line from a given conic section whose plane does not contain the straight line, and obtain the result when the conic is a circle.

Solution by K. Satyanarayana and K. Srinivasa Raghavan.

Case (i) Central Conic: Referred to its principal axes, let its equation be

$$ax^2 + by^2 = 1 \qquad \ldots \qquad \ldots \qquad (1)$$

and let the line cut the plane at an angle α at (h, k) and let the orthogonal projection of the line be $x = h + r\cos\beta$, $y = k + r\sin\beta$, r being the distance of any point on it from (h, k). Then any point P on the line is

$$x = h + r\cos\beta\,;\ y = k + r\sin\beta\,;\ z = r\tan\alpha.$$

Let Q (x, y) on (1) be nearest to P, so that

$$[(h + r\cos\beta - x)^2 + (k + r\sin\beta - y)^2 + r^2\tan^2\alpha]$$

is least: the condition for which is easily seen to be

$$\frac{h + r\cos\beta - x}{ax} = \frac{k + r\sin\beta - y}{by}. \qquad \ldots \qquad (2)$$

Any point on PQ is

$$\left.\begin{aligned} X &= \frac{h + r\cos\beta + \lambda x}{\lambda + 1}, \\ Y &= \frac{k + r\sin\beta + \lambda y}{\lambda + 1}, \\ Z &= \frac{r\tan\alpha}{\lambda + 1}. \end{aligned}\right\} \qquad \ldots \qquad \ldots \qquad (3)$$

Elimination of x, y, r, λ from (1), (2), (3) leads to the required result.

Case (ii) Parabola: Take its equation as $y^2 = 4ax$. ... (1′)

Then (2) becomes $\dfrac{h + r\cos\beta - x}{2a} + \dfrac{k + r\sin\beta - y}{y} = 0$. .. (2′)

Elimination from (1′), (2′), (3) leads to the result.

Case (iii) Circles : Q may be taken as $(a\cos\theta,\ a\sin\theta)$, k may be made zero. Here (2) reduces to

$$\frac{h + r\cos\beta}{\cos\theta} = \frac{r\sin\beta}{\sin\theta} = \mu\ (\text{say}), \qquad \dots \quad (2'')$$

and (3) becomes

$$X = \frac{h + r\cos\beta + \lambda a\cos\theta}{\lambda+1} = \left(\frac{\mu+\lambda a}{\lambda+1}\right)\cos\theta, \quad \dots \quad (3'')$$

$$Y = \frac{r\sin\beta + \lambda a\sin\theta}{\lambda+1} = \left(\frac{\mu+\lambda a}{\lambda+1}\right)\sin\theta, \quad \dots \quad (4'')$$

$$Z = \frac{r\tan\alpha}{\lambda+1}. \quad \dots \quad \dots \quad \dots \quad \dots \quad (5'')$$

From (2″) and (3″)

$$\frac{h + r\cos\beta}{X} = \frac{r\sin\beta}{Y} = \frac{h\sin\beta}{X\sin\beta - Y\cos\beta}; \qquad \dots \quad (6'')$$

(5″) now gives

$$\lambda + 1 = \frac{hy\tan\alpha}{Z(X\sin\beta - Y\cos\beta)}; \quad \dots \quad \dots \quad (7'')$$

(2″) gives

$$\left.\begin{aligned}\mu\cos\theta &= \frac{h\sin\beta}{X\sin\beta - Y\cos\beta}\cdot X\\ \mu\sin\theta &= \frac{h\sin\beta}{X\sin\beta - Y\cos\beta}\cdot Y\end{aligned}\right\}\text{by using } (6''). \quad \dots \quad (8'')$$

(3″) and (4″) become

$$(\lambda+1)X - \mu\cos\theta = \frac{h}{Z}\left(\frac{Y\tan\alpha - Z\sin\beta}{X\sin\beta - Y\cos\beta}\right)X = \lambda a\cos\theta. \quad (9'')$$

$$(\lambda+1)Y - \mu\sin\theta = \frac{h}{Z}\left(\frac{Y\tan\alpha - Z\sin\beta}{X\sin\beta - Y\cos\beta}\right)Y = \lambda a\sin\theta. \quad (10'')$$

Squaring (9″) and (10″) and taking 'λ' from (7″) the surface traced out is

$$h^2(Y\tan\alpha - Z\sin\beta)^2(X^2+Y^2) = [hY\tan\alpha - Z(X\sin\beta - Y\cos\beta)]^2a^2.$$

Question 1239.

(R. VAIDYANATHASWAMY):—If P, Q, R, S be four points in an Argand diagram, P′, Q′, R′, S′ their inverses with respect to a real or imaginary circle, show that the cross ratios (PQRS) and (P′Q′R′S′) are conjugate complex numbers.

Solution by S. Audinarayanan.

Suppose the points P, Q, R, S to represent the complex numbers x_1+iy_1, x_2+iy_2, x_3+iy_3 and x_4+iy_4 in the Argand diagram. Now the cross-ratio (PQRS)

$$= \frac{[\overline{x_1-x_2}+i\,\overline{y_1-y_2}]\,[\overline{x_3-x_4}+i\,\overline{y_3-y_4}]}{[\overline{x_1-x_4}+i\,\overline{y_1-y_4}]\,[\overline{x_3-x_2}+i\,\overline{y_3-y_2}]}$$

$$= \frac{(a+bi)(c+di)}{(e+fi)(g+hi)}$$

$$= \frac{(ac-bd)+i(bc+ad)}{(eg-hf)+i(fg+ch)} = \frac{x+iy}{z+iw} \text{ say,}$$

$$= \frac{xz+yw+i(yz-xw)}{z^2+w^2}.$$

We know that the inverse of a point is the reflection of the geometrical inverse with respect to the x-axis. The geometrical inverse of P, Q, R, S are points in the Argand diagram representing numbers

$$\frac{k^2}{x_1+iy_1},\ \frac{k^2}{x_2+iy_2},\ \frac{k^2}{x_3+iy_3},\ \frac{k^2}{x_4+iy_4}.$$

The numbers represented by the true inverse points will have the sign of their x's opposite to those of the geometrical inverse points.

Hence the numbers represented by the inverse points P′, Q′, R′, S′ are

$$\frac{k^2}{-x_1+iy_1},\ \frac{k^2}{-x_2+iy_2},\ \frac{k^2}{-x_3+iy_3},\ \frac{k^2}{-x_4+iy_4}.$$

Now the cross ratio (P′Q′R′S′)

$$= \frac{\left(\frac{1}{x_2-iy_2}-\frac{1}{x_1-iy_1}\right)\left(\frac{1}{x_4-iy_4}-\frac{1}{x_3-iy_3}\right)}{\left(\frac{1}{x_2-iy_2}-\frac{1}{x_3-iy_3}\right)\left(\frac{1}{x_4-iy_4}-\frac{1}{x_1-iy_1}\right)}$$

$$= \frac{[\overline{x_1-x_2}-i\,\overline{y_1-y_2}]\,[\overline{x_3-x_4}-i\,\overline{y_3-y_4}]}{[\overline{x_3-x_2}-i\,\overline{y_3-y_2}]\,[\overline{x_1-x_4}-i\,\overline{y_1-y_4}]}$$

$$= \frac{(a-bi)(c-di)}{(g-hi)(e-fi)}$$

$$= \frac{(ac-bd)-i(bc+ad)}{(eg-hf)-i(fg+ch)} = \frac{x-iy}{z-iw}$$

$$= \frac{xz+yw-i(yz-xw)}{z^2+w^2}$$

= conjugate of (PQRS).

Thus it is established that the cross ratios (PQRS) and (P′Q′R′S′) are conjugate complex numbers.

Question 1251.

(C. KRISHNAMACHARI AND M. BHIMASENA RAO):—Let $[A_r, A_n]$ stand for the persymmetric determinant

$$\begin{vmatrix} A_r, & A_{r+1}, & A_{r+2}, & \ldots\ldots & A_n \\ A_{r+1}, & A_{r+2}, & A_{r+3}, & \ldots\ldots & A_{n+1} \\ A_{r+2}, & \ldots\ldots & & & \\ \ldots\ldots & & & & \\ A_n, & A_{n+1}, & & & A_{2n-r} \end{vmatrix}$$

Then show that:

(*a*) If $A_s = 1.3.5 \ldots\ldots (2s-1)$,
$[A_r, A_n] = A_r . A_{r+1} \ldots\ldots A_n . 2^{n-r} . 4^{n-r-1} . 6^{n-r-2} \ldots\ldots (2n-2r)$.

(*b*) If $A_s = s\,!$,
$[A_r, A_n] = A_r A_{r+1} \ldots\ldots A_n \cdot 1!.2!.3! \ldots\ldots (n-r)!$

(*c*) If $A_s = m(m+1)(m+2) \ldots\ldots (m+s-1)$,
$[A_r, A_n] = A_r . A_{r+1} \ldots\ldots A_n . 1!.2!.3! \ldots\ldots (n-r)!$

(*d*) The value of the determinant

$$\begin{vmatrix} 1 & 1 & \ldots\ldots & 1 \\ m+r & m+r+1 & \ldots\ldots & m+n \\ (m+r)_2 & (m+r+1)_2 & \ldots\ldots & (m+n)_2 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots\ldots \\ (m+r)_{n-r} & (m+r+1)_{n-r} & \ldots\ldots & (m+n)_{n-r} \end{vmatrix}$$

is independent of m, where x_p denotes
$$x(x+1)(x+2) \ldots\ldots (x+p-1).$$

Solution by Martyn Thomas.

The mode of evaluation of the several determinants is the same for all, namely, by first rewriting them with the elements of the several columns beginning with those of the first column, so that after removal of the common factors $A_r, A_{r+1} \ldots\ldots A_n$, and changing every row but the first one into the difference between that row and its preceding one, the common factors may be removed again, and the process may be repeated, the order of the determinants being lowered thereby.

(*a*) $A_r = 1.3.5 \ldots\ldots (2r-1)$.

$$\therefore \Delta = \begin{vmatrix} A_r, & (2r+1)A_r, & (2r+1)(2r+3)A_r, & \ldots\ldots \; (2r+1)(2r+3) \ldots\ldots (2n-1)A_r \\ A_{r+1}, & (2r+3)A_{r+1}, & (2r+3)(2r+5)A_{r+1}, & \ldots\ldots \; (2r+3)(2r+5) \ldots\ldots (2n+1)A_{r+1} \\ A_n, & (2n+1)(2n+1)A_n, & \ldots & (2n+1) \ldots\ldots (4n-2r-1)A_n \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\begin{vmatrix} 1, & (2r+1), & \ldots\ldots\ldots & \\ 0, & 2, & (2r+3)4, & (2r+3)\ldots(2n-1)(2n-2r) \\ \ldots & \ldots & \ldots & \ldots \\ 0, & 2, & (2n+1)4, & (2n+1)\ldots\ldots(2n-2r) \end{vmatrix}$$

$$=(A_rA_{r+1}..A_n).2.4.6..(2n-2r)\begin{vmatrix} 1, & 2r+3, & \ldots\ldots \\ 0, & 2, & (2r+5)4,\ldots(2r+5)\ldots(2n-2r-2) \\ 0, & 2, & (2n+3)4,\ldots(2n+3)\ldots(2n-2r-2) \end{vmatrix}$$

$$= (A_rA_{r+1}\ldots\ldots A_n)\,[2.4.6\ldots\ldots(2n-2r)]\,[2.4.6\ldots\ldots(2n-2r-2)]\times$$
$$[2.4.6\ldots\ldots(2n-2r-4)]\ldots\ldots[2.4.6]\,[2.4]\,[2]$$

$$= (A_rA_{r+1}\ldots\ldots A_n).\,2^{n-r}.\,4^{n-r-1}.\,6^{n-r-2}\ldots\ldots(2n-2r-2)^2(2n-2r).$$

(*b*) Here $A_r = r!$.

$$\Delta=\begin{vmatrix} A_r, & (r+1)A_r, & (r+1)(r+2)A_r, & \ldots(r+1)(r+2)\ldots n.A_r \\ A_{r+1}, & (r+2)A_{r+1}, & (r+2)(r+3)A_{r+1}, & \ldots(r+2)(r+3)\ldots(n+1)A_r \\ \ldots & \ldots & \ldots & \ldots \\ A_n, & (n+1)A_n, & \ldots\ldots & (n+1)(n+2)\ldots(2n-r)A_{r+1} \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\begin{vmatrix} 1, & r+1, & \ldots\ldots & \\ 0, & 1, & (r+2)2, & \ldots\ldots(r+2)\ldots\ldots n(n-r) \\ 0, & 1, & (n+1)2, & \ldots\ldots(n+1)\ldots\ldots(n-r) \end{vmatrix}$$

$$= (A_rA_{r+1}\ldots A_n)\,[1.2.3\ldots(n-r)]\,[1.2.3\ldots(n-r-1)]\times$$
$$[1.2.3\ldots(n-r-2)]\ldots[1.2.3]\,[1.2]\,[1]$$
$$= (A_rA_{r+1}\cdots A_n)(n-r)!\,(n-r-1)!\ldots 3!\,2!\,1!.$$

(c) Here $A_r = m(m+1)(m+2)\cdots\cdots(m+r-1)$.

$$\Delta=\begin{vmatrix} A_r, (m+r)A_r, (m+r)(m+r+1)A_r\ldots\ldots, \\ (m+r)\ldots\ldots(m+n-1)A_r \\ A_{r+1}, (m+r+1)A_{r+1}, (m+r+1)(m+r+2)A_{r+1}\ldots, \\ (m+r+1)\ldots\ldots(m+n)A_{r+1} \\ \ldots\ldots \\ A_n, (m+n)A_n, \ldots\ldots \end{vmatrix}$$

$$=(A_rA_{r+1}\ldots A_n)\begin{vmatrix} 1, & m+r, & \ldots\ldots & \\ 0 & 1, & (m+r+1)2\ldots\ldots & (m+r+1)\ldots(n-r) \\ \ldots & & & \\ 0, & 1, & (m+n)2 & \ldots\ldots \end{vmatrix}$$

$$=(A_rA_{r+1}\cdots A_n)[1.2.3\ldots(n-r)]\,[1.2.3\cdots(n-r-1)]\ldots[1.2.3]\,[1.2]\,[1]$$
$$=(A_rA_{r+1}\cdots A_n)(n-r)!\,(n-r-1)!\cdots 3!\,2!\,1!.$$

(*d*) Here $(m+r)_p = (m+r)(m+r+1)\cdots$ to p factors.

$$\Delta = \begin{vmatrix} 1, & 1 & \cdots\cdots\cdots & 1 \\ m+r, & m+r+1 & & m+n \\ (m+r)(m+r+1), & (m+r+1)(m+r+2), & \cdots\cdots & (m+n),(m+n+1) \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ (m+r)\cdot\cdot(m+n-1), & (m+r+1)\cdots(m+n), & \cdots & (m+n)\cdot\cdot(m+2n-r-1) \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0, & \cdots\cdots\cdots \\ m+r, & 1, & 1, & \cdots\cdots\cdots \\ 1 & (m+r+1)2, & (m+r+2), & \cdots\cdots\cdots \\ 1 & & & \\ 1 & & & \\ 1 & & & \\ 1 & (m+r+1)\ldots.(n-r), & (m+r+2)\ldots.(n-r), & \end{vmatrix}$$

$= [1.2.3 \cdots\cdots (n-r)]\,[1.2.3 \cdots\cdots (n-r-1)] \cdots\cdots [1.2.3][1.2][1]$

$= (n-r)!\,(n-r-1)! \quad \cdots\cdots 3!\,2!\,1!,$

which is independent of m.

Question 1259.

(P. V. SESHU IYER) :—(1) Show that the values of x for which the function $u = xf(x)$ is a maximum are the abscissæ of the points where $y = f(x)$ touches a member of the hyperbola $xy = c$.

(2) If $y = f(x)$ touches a member of the family $xy = c$ at a point whose abscissa is x, show that the curve $y = f(x) - \frac{k}{x}$, touches another of the family at a point of the same abscissa x, and find the constant c of that other member.

Solution by P. Krishnamachari, S. Audinarayanan, I. Totadri Iyengar, C. Ranganathan and V. A. Mahalingam.

(1) The values of x for which the function $u = xf(x)$ is a maximum are given by the equation $\quad \dfrac{du}{dx} = 0,$

i.e., by
$$f(x) + xf'(x) = 0. \qquad \ldots \qquad \ldots \quad (1)$$

The abscissæ of the points of intersection of $y = f(x)$ with $xy = c$ are given by
$$xf(x) = c, \quad \ldots \quad \ldots \quad \ldots \quad (2)$$

and if the two curves should also touch at this point
$$f'(x) = -\frac{c}{x^2}. \qquad \ldots \qquad \ldots \quad (3)$$

The eliminant of (2) and (3) gives the abscissæ of the point of contact of $y = f(x)$ with the family of curves $xy = c$; and this is

$$x\,[f(x) + xf'(x)] = 0.$$

Rejecting the factor x, we see that the first part follows.

(2) If c_1 is the 'c' of the first member and c_2 that of the second, then from the above $\quad x\,f(x) = c_1;$

and in the latter case $\quad x\,f(x) - k = c_2, \quad \ldots \quad \ldots \quad (4)$

and $\quad f'(x) - \dfrac{k}{x^2} = -\dfrac{c_2}{x^2}. \quad \ldots \quad \ldots \quad (5)$

Eliminating c_2 from (4) and (5), we get $f(x) + xf'(x) = 0$, which shows that x can be the abscissa of the point of contact of $y = f(x) - \frac{k}{x}$ with a member of the family of rectangular hyperbolas.

From (4), we see that $c_2 = c_1 - k$, which gives the 'c' of the latter member in terms of that of the former.

Question 1265.

(A. A. KRISHNASWAMY IYENGAR) :—Show that every odd number can be expressed as the sum of seven squares (excepting 1, 3, 5, 9, 11, 17).

Solution by I. Totadri Iyengar.

LEMMA :—If $3\,M = \alpha^2 + \beta^2 + \gamma^2 + \delta^2$ (in which of course either only one of them say α or all of them should be multiples of 3), then $M = \alpha_1^2 + \beta_1^2 + \gamma_1^2 + \delta_1^2$,

where
$$\left.\begin{aligned} \alpha_1 &= \frac{\sqrt{\beta^2} + \sqrt{\gamma^2} + \sqrt{\delta^2}}{3} \\ \beta_1 &= \frac{\alpha + \sqrt{\beta^2} - \sqrt{\gamma^2}}{3} \\ \gamma_1 &= \frac{\alpha + \sqrt{\gamma^2} - \sqrt{\delta^2}}{3} \\ \delta_1 &= \frac{\alpha + \sqrt{\delta^2} - \sqrt{\beta^2}}{3}. \end{aligned}\right\}$$

To the absolute value of the radical that sign is to be attached which makes
$$\sqrt{\beta^2} \equiv \sqrt{\gamma^2} \equiv \sqrt{\delta^2} \pmod{3}.$$

Let the given number n be of the form $4\lambda + 1$, and let p be any prime contained in the form $3^{2x+1}\,n - 2$.

Evidently $p \equiv 1 \text{ mod. } 4.$

$\therefore$ $p = \alpha^2 + \beta^2.$

Then $3^{2x+1}.n = \alpha^2 + \beta^2 + 1^2 + 1^2.$

So that, by the Lemma, we can express n as the sum of 4 squares.

Let finally therefore $n = a^2 + b^2 + c^2 + d^2$ of which if a, b, c, d are not all multiples of 3, a is the only multiple of 3.

Then a^2 is of the form $4n + 1$ and hence can be split up as above into the sum of 4 squares, say

$$a^2 = e^2 + f^2 + g^2 + h^2.$$

So we finally get

$$n = b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2$$

as the solution.

If n is of the form $4\lambda + 3$, we take p to be the prime contained in the form $3^{2x}.n-2$ and proceed as before.

In the special cases mentioned some of the numbers b, c, d, etc., vanish.

N. B. With certain exceptions the same theorem holds also in the case of the even numbers, (*Vide* Sylvester: *Collected Works*, Vol. II, p. 102.) In general, we can say that all numbers can be expressed as the sum of $(3n + 1)$ squares.

Question 1271.

(K. SATYANARAYANA):—(i) p being any odd prime, prove that

$$2^{p-1} - 1 \not\equiv 0 \text{ mod. } (p^2.)$$

Hence or otherwise, establish that the numerator of

$$\left\{1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots\dots\dots + (-1)^{\frac{p+1}{2}} \frac{1}{\frac{1}{2}(p-1)}\right\} \not\equiv 0 \text{ mod. } p.$$

(ii) p being a prime greater than 3, prove that,

$$\left\{(p-1)! - (-1)^{\frac{p-1}{2}} \left[2^{p-1}\left(\frac{p-1}{2}\right)!\right]^2\right\} \equiv 0 \text{ mod. } p^3$$

Solution by I. Totadri Iyengar.

(i) Let $p = 2n + 1$. If it be possible, let

$$2^{p-1} \equiv 1 \text{ mod. } p^2.$$

Then

$$2^{2p-2} = 2^{4n} \equiv 1 \text{ mod. } p^2$$

or

$$\frac{2^{4n} - 1}{n} \equiv 0 \text{ mod. } p.$$

Hence from Q, 1247 already solved [*J. I. M. S.*, XV, p 60.]

$$a \equiv (-1)^n b \text{ mod. } p.$$

i. e.,

$$\frac{(2n)! + 1}{p} \equiv (-1)^n . \frac{n! + (-1)^n}{p} \text{ mod. } p$$

i.e.,

$$(2n)! \equiv (-1)^n n! \text{ mod. } p^2$$

Hence $(p-1)(p-2)\,..\,(p-n) \,.\, n! \equiv (-1)^n \, n! \text{ mod. } p^2.$

$$\therefore (-1)^n \left\{ n! - n!\, p \left(1 + \tfrac{1}{2} + \tfrac{1}{3} + . + \frac{1}{n}\right) \right\} \equiv (-1)^n \text{ mod. } p^2.$$

Now from the solution to Q. 1197, since $2^{4n} - 1 \equiv 0 \text{ mod. } p^2$

$$\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n}\right) n! \equiv 0 \text{ mod. } p.$$

$\therefore$ $\quad n! \equiv 0 \text{ mod. } p^2$ which is impossible.

Hence

$$2^{p-1} - 1 \not\equiv 0 \text{ mod. } p^2.$$

Now

$$2^{p-1} - 1 = (p-1) + \frac{(p-1)\,(p-2)}{1\,.\,2} + \ldots\ldots$$

$$\frac{(p-1)\,(p-2)}{1\,.\,2} + (p-1) + 1$$

$$= p + p \,.\, \frac{p-1}{1\,.\,2} + p\,\frac{(p-1)\,(p-2)}{1\,.\,2\,.\,3} + \ldots$$

on adding the 1st and the last; the 2nd and the last but one, etc.

Simplifying we have the numerator of

$$\left(1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \ldots\ldots + (-1)^{\frac{p+1}{2}} \frac{1}{n}\right) \not\equiv 0 \text{ mod. } p.$$

(ii) It is well-known from Lagrange that the numerator of

$$\left(1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{p-1}\right) \equiv 0, \text{ mod. } p\,;$$

also that of

$$1 + \frac{1}{1\,.\,2} + \frac{1}{1\,.\,3} + \ldots\ldots + \frac{1}{2\,.\,3} + \,..\, \equiv 0, \text{ mod. } p.$$

Hence on squaring the first and subtracting twice the second, we get

$$\left\{1+\frac{1}{2^2}+\frac{1}{3^2}+\ .\ +\frac{1}{(p-1)^2}\right\} \equiv 0 \text{ mod. } p.$$

But

$$1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ \ldots\ldots\ \frac{1}{(p-1)^2}=\left\}1-\frac{1}{(p-1)^2}\right\}$$

$$+\left\{\frac{1}{2^2}-\frac{1}{(p-2)^2}\right\}+\left\{\frac{1}{3^2}-\frac{1}{(p-3)^2}\right\}+\ldots\ldots$$

$$\equiv 0 \text{ mod. } p.$$

$$\therefore \qquad 1+\frac{1}{3^2}+\frac{1}{5^2}+\ \ldots\ldots\ +\frac{1}{(p-2)^2} \equiv 0 \text{ mod. } p.$$

From the solution to Q. 1247

$$2^{2n}\,.\,2n\,! \equiv (-1)^n\,1^2\,.\,3^2\ \ldots\ldots\ (p-2)^2 \text{ Nr. of } \left(1+\frac{1}{3^2}+\frac{1}{5^2}+\ \ldots\ldots\right.$$

$$\left.+\frac{1}{p-2^2}\right)^2 \text{mod. } p^3.$$

$$\equiv (-1)^n\,1^2\,.\,3^2\ \ldots\ldots\ (p-2)^2, \text{ mod. } p, \text{ from the above.}$$

Hence just as in Q. 1247, it follows that

$$2n\,!-(-1)^n\,2^{4n}\,(n\,!)^2 \equiv 0, \text{ mod. } p^3,$$

which is the result required.

Questions for Solution.

1336. (MARTYN THOMAS):—An infinite straight line occupying the positive part of the axis of z attracts according to the law of nature. Show, from elementary considerations, that its potential is a function of $(r-z)$, and hence, or otherwise, prove that the potential is $A \log (r-z)+B$, where A and B are constants.

1337. (A. NARASINGA RAO):—a_1, a_2 are a pair of straight lines through A and b_1, b_2 another pair through B. Show that

(i) through a given point x can be drawn in general a pair of lines x_1, x_2 in two different ways so that the quadrilaterals $a_1 a_2 x_1 x_2$ and $b_1 b_2 x_1 x_2$ are circumscribed to circles.

(ii) if c_1 c_2 be a third pair of lines through C, and if the quadrilaterals $a_1 a_2 x_1 x_2$, $b_1 b_2 x_1 x_2$, $c_1 c_2 x_1 x_2$ are circumscribed to circles, the common point X of x_1 and x_2 must lie on one of four onics in the plane.

1338. SELECTED—(Prof. CAYLEY):—Show that the surfaces

$$xyz = 1, \quad yz + zx + xy + x + y + z + 3 = 0$$

intersect in two distinct cubic curves; and find the equations of the cubic cones which have their vertices at the origin and pass through these curves respectively.

1339. (R. VAIDYANATHASWAMY):—A quadratic form $\Sigma a_{ij}\, x_i\, x_j$ $(i, j = 1, 2, \ldots n)$ is such that

$$a_{ji} a_{jk} a_{ik} = 0, \quad a_{ij} = a_{ji}$$

Show that by permuting the variables it can be reduced to the form

$$2\,(a_{12}\, x_1\, x_2 + a_{23}\, x_2\, x_3 + a_{34}\, x_3\, x_4 + \ldots\ldots + a_{n-1,\,n}\, x_{n-1}\, x_n.)$$

1340. (R. GOPALASWAMI):—A conic cuts the sides of a quadrilateral at eight points. If two of these points lying on different sides be joined to meet the two other sides, two new points are obtained. Show that the eight new points obtained (by choosing four pairs in any manner whatever) lie on a conic.

1341. (R. GOPALASWAMI):—A_r $(r = 1, 2, 3, 4)$ are four points on the sides of a quadrilateral. A conic through A_r cuts the sides at four other points B_r. A conic through B_r cuts them at C_r and so on the tetrads D_r, E_r, F_r are obtained. Show that any two of the tetrads A_r, B_r, C_r, D_r, E_r, F_r lie on a conic. The conic $(A_r B_r)$ meets $(D_r E_r)$ at points P_r. $(B_r C_r)$ meets $(E_r F_r)$ at Q_r. $(C_r D_r)$ meets $(F_r A_r)$ at R_r. Show that the twelve points P_r, Q_r, R_r lie on a conic.

The above is an extension of Pascal's theorem. Extend it further.

1342. (V. RAMASWAMI AIYAR):— Prove that

$$\begin{vmatrix} 0 & x & x' & 0 & a & a' \\ 0 & x' & x & 0 & a' & a \\ y' & 0 & y & b' & 0 & b \\ y & 0 & y' & b & 0 & b' \\ z & z' & 0 & c & c' & 0 \\ z' & z & 0 & c' & c & 0 \end{vmatrix} = \begin{vmatrix} x^2 - x'^2, & y^2 - y'^2, & z^2 - z'^2 \\ a^2 - a'^2, & b^2 - b'^2, & c^2 - c'^2 \\ ax - a'x', & by - b'y', & cz - c'z' \end{vmatrix}$$

1343. (V. RAMASWAMI AIYAR):—Given a triangle ABC, if P be a variable point on a fixed diameter of its circumcircle that, we know its pedal circle passes through a fixed point Ω. Prove that the rectangular hyperbola passing through P and the feet of the perpendiculars drawn therefrom to the sides cuts the pedal circle of P again at the point diametrically opposite to Ω.

1344. (V. TIRUVENKATACHARYA):—Show that if $|q| < 1$,

(i) $(1-q^4)^2(1-q^8)(1-q^{12})^2(1-q^{16})\ldots\ldots = 1 - 2q^4 + 2q^{16} - \ldots$

(ii) $[(1-q^2)(1-q^6)(1-q^{10})\ldots\ldots]^{-1} = (1+q^2)(1+q^4)(1+q^6)\ldots.$

1345. (F. H. V. GULASEKHARAN):—If each of the ratios

$$(x_p,\ y_p,\ z_p,\ t_p),\ (x_q,\ y_q,\ z_q,\ t_q)$$

satisfies simultaneously the equations

$$\left.\begin{aligned} u_1x + u_2y + u_3z + u_4t &= 0 \\ v_1x + v_2y + v_3z + v_4t &= 0 \end{aligned}\right\}$$

and if

$$S_{pq} = \Sigma\, a x_p x_q + \Sigma f(y_p z_q + y_q z_p),$$

the summation extending to the variables x, y, z, t and the co-efficients $a, b, c, d, f, g, h, l, m, n$; and if S'_{pq} denotes a similar function with co-efficients a', b', c' etc., prove that

$$\frac{S_{pp}S_{qq} - S_{pq}^2}{\Delta} = \frac{S_{pp}S'_{qq} + S_{qq}S'_{pp} - 2S_{pq}S'_{pq}}{\phi} = \frac{S'_{pp}S'_{qq} - S'^2_{pq}}{\Delta'},$$

where Δ, ϕ, Δ' are the co-efficients of k^2, k, k^0 in the equation

$$\begin{vmatrix} ka + a', & kh + h', & kg + g', & kl + l', & u_1, & v_1 \\ kh + h', & kb + b', & kf + f', & km + m', & u_2, & v_2 \\ kg + g', & kf + f', & kc + c', & kn + n', & u_3, & v_3 \\ kl + l', & km + m', & kn + n', & kd + d', & u_4, & v_4 \\ u_1, & u_2, & u_3, & u_4, & 0, & 0 \\ v_1, & v_2, & v_3, & v_4, & 0, & 0 \end{vmatrix} = 0.$$

Explain the results geometrically.

[The extension to the case of n variables satisfying $(n-2)$ linear equations is obvious.]

Notes and Questions.

Did the Greeks Solve the Quadratic Equation?

One frequently meets with the statement that the ancient Greeks solved the quadratic equation. For instance, on page 250 of Volume I of Pascal's *Repertorium der höheren Mathematik*, 1910, it is stated that the solution of the quadratic equation was known to the Greeks. On page 126 of Volume I of Smith's *History of Mathematics*, 1923, we find the stronger statement "that the general quadratic as we know it to-day was thus fully mastered by the Greek mathematicians." It seems to the writer of the present note that it would be more correct to say that the ancient Greeks found formulas by means of which the quadratic equation can be solved, but they did not advance far enough in their algebraic work to solve the general quadratic equation in one unknown. By means of the formulas they found one root of many special quadratic equations.

While the expression 'solution of the algebraic equation of the second degree' is not very definite, the student naturally understands by it at least as much knowledge about this equation as is expected of him when he is examined on this equation in a course in elementary algebra. For instance, one fundamental fact about this equation is that it always has a root. Since the Greeks knew nothing about imaginary numbers they could not have known this fact. Another fundamental fact is that the first member of the equation $ax^2 + bx + c = 0$ can always be resolved into two linear factors. As the Greeks were ignorant of the factor theorem, even in the elementary form as applied to the general quadratic expression in one unknown, they could not have known this fact even in the case when the two roots are positive and rational. As the Greeks knew nothing about equal roots they, of course, did not know anything about a general criterion relating to the equality of the roots of the quadratic equation. They also knew nothing about the relations between the co-efficients and the roots of this equation.

It might be said that when the student comes across the statement that the Greeks 'mastered the quadratic equation,' he should not be misled thereby since people who did not know even *negative* numbers could not have 'mastered' the quadratic equation in the strict modern sense. There is no doubt, however, that the above statement has given rise to much confusion on the part of the reader. Moreover the fact that the Greeks

found general formulas for the solution of the quadratic equation, which were not fully understood by them or by other mathematicians for a period of one thousand years after their discovery, gives some insight into the extremely slow historical development of the subject. More than two thousand years may have passed from the time of this discovery until imaginary numbers were fully accepted as legitimate numbers through the labours of Caspar Wessel, J. R. Argand, C. F. Gauss, W. R. Hamilton, and many others.

The great difference between the casual discovery of a formula and the true understanding of the full significance of the formula is perhaps nowhere more clearly exhibited than in the history of the solution of the quadratic equation. Hence it seems desirable that the student's attention should be directed to this point. He will thereby learn a very fundamental fact in the history of mathematics. For instance, he will see that it is not quite accurate to say that the general cubic equation was solved algebraically by the Italian mathematicians during the first half of the sixteenth century, though it is true that a formula, which is now commonly called Cardan's formula, was discovered during this period. More than two hundred years passed from the time when this formula appeared in the *Ars Magna* (1545), before it was pointed out by D'Alembert in the *Encyclopédie ou dictionnaire raisonnée des sciences*, Volume 2, page 736, that this formula represents not only one root but the three roots of the cubic equation.

Just as in the case of the quadratic equation, the formula for the general solution of the algebraic equation of the third degree was discovered a long time before this formula was fully understood. In both cases, the formulas were first used to find *one root* of special equations. In both cases, those who discovered them knew comparatively little about their general significance. It is true that each of them took a very important step towards the solution of the equation in question, but it was only a step. This step had to be followed by the work of many others before the 'solution'—in the modern sense of this term—was completed. A knowledge of the fact that the labours of many mathematicians, who lived at different times and in different countries, are reflected in some of the most fundamental results with which the student of elementary mathematics has to deal should help him to realize and appreciate the value of modern co-operative efforts as exhibited by the modern mathematical societies and their periodicals.

From what precedes it is easy to infer that our answer to the question which is the subject of the present note is that the Greeks did not

solve the quadratic equation. We would add that the Greeks, the Indians and the Arabs, taken together, did not solve this equation, although each of them made important contributions towards the solution. In fact, contributions towards the solution as it is now presented to students of elementary mathematics extend into the nineteenth century and include the work of those who laid a solid foundation for the use of the negative and the general complex numbers. The more advanced student will find that the modern theory of irrational numbers and the Galois theory of equations throw additional light on this solution, though the term solution need not imply such deep insight. It seems, however, clear that those who considered many quadratic equations as *insolvable* should not be regarded as having 'solved' the general quadratic equation, unless the term *solved* is used in a restricted sense.

G. A. MILLER.

On the Eight Spheres touching the faces of a Tetrahedron.

With the usual convention of signs, the centres of the eight spheres are the points $(1, \pm 1, \pm 1, \pm 1)$. Denote these by I, I_1, I_2. I_3, I_4, I_a, I_b. I_c. Let the corresponding radii be r, r_1, r_2, r_3, r_4, r_a, r_b, r_c.

We have

$$\text{vol. IBCD} + \text{vol. ICDA} + \text{vol. IDAB} + \text{vol. IABC} = \text{vol. ABCD}.$$

$$\therefore \quad r(A_1 + A_2 + A_3 + A_4) = 3 \text{ vol. ABCD}.$$

$$\therefore \qquad \frac{1}{r} = \frac{2}{3} \cdot \frac{S}{V}, \qquad \dots \qquad \dots \quad (1)$$

where A_1, A_2, A_3, A_4 are the areas of faces opposite to A, B, C, D and

$$2S = A_1 + A_2 + A_3 + A_4,$$

and

$$V = \text{vol. ABCD}.$$

Similarly $r_1^{-1} = \frac{2}{3} \cdot \frac{(S-A_1)}{V}$, $r_2^{-1} = \frac{2}{3} \cdot \frac{(S-A_2)}{V}$, etc. ... (1.1)

and

$$r_a^{-1} = \frac{2}{3} \cdot \frac{(S - A_1 - A_2)}{V}, \text{ etc.} \qquad \dots (1.2)$$

From relations (1.1), (1.2), we deduce

$$2 \cdot r^{-1} = r_1^{-1} + r_2^{-1} + r_3^{-1} + r_4^{-1}; \qquad \dots (1.3)$$

$$2 \cdot r_a^{-1} = r_1^{-1} + r_2^{-1} - r_3^{-1} - r_4^{-1}, \text{ \&c.} \qquad \dots (1.4)$$

2. Let the spheres touch the face BCD opposite to A in α, α_1, α_2, α_3, α_4, α_a, α_b, α_c; the face opposite to B in β, β_1, β_2, ... β_c; and so on. Further, let the plane angles at the vertices A, B, C, D be denoted by letters A, B, C, D with suffixes 1, 2, 3, 4, according as they lie in the faces opposite A, B, C, D. Thus, angle BAD is A_3. With this notation, we easily see that triangles $A\beta C$ and $A\delta C$ are congruent.

$$\therefore \quad C\hat{A}\delta = C\hat{A}\beta.$$

Similarly $$D\hat{A}\beta = D\hat{A}\gamma,$$

$$B\hat{A}\gamma = B\hat{A}\delta.$$

Hence $$C\hat{A}\beta = C\hat{A}\delta = (\sigma_1 - A_3), \text{ etc.}$$

Similarly $$D\hat{A}\beta_1 = \sigma_1 - A_3, \text{ etc.}$$

Thus it will be seen that β, β_1, *are isogonal conjugates with respect to the face ACD*. Similarly the six points β_2, β_3, β_4, β_a, β_b, β_c can be divided into pairs of isogonal conjugate points. Thus (β, β_1), (β_2, β_a), (β_3, β_b), (β_4, β_c), are isogonal pairs for the triangle ACD.

Corresponding properties hold for the remaining faces.

3. The perpendicular IP on CD is perpendicular to the plane I_1CD; for, the plane ICD is perpendicular to the plane I_1CD.

$\therefore$ IP is perpendicular to I_1P.

Thus II_1 subtends a right angle at the six feet of perpendiculars from I, I_1 on BC, CD, DB and so they lie on a circle.

Similarly, it can be proved that the 24 feet of the perpendiculars from the centres of the eight spheres on BC, CD, DB lie six by six on four circles.

Further, since IP is perpendicular to CD and also $I\alpha$ is perpendicular to the plane BCD, αP is perpendicular to CD. Thus the feet of the perpendiculars above coincide with those from α, α_1, α_2, ... α_a.....α_c on the sides of the triangle BCD.

4. From § 2 we see that (α, α_1), (α_2, α_a), (α_3, α_b), (α_4, α_c) are the foci of conics touching the sides of $\triangle BCD$; and that the circles of § 3 are the auxiliary circles of these conics.

If X is the foot of the $\perp^r$ from A on BCD, then since A, I, I_1 are collinear, their projections X, α, α_1, are collinear. Similarly, the joins of $\alpha_2\alpha_a$, $\alpha_3\alpha_b$, $\alpha_4\alpha_c$ pass through the same point X.

5. Since the planes IBC, I_1BC, bisect the angle between the planes DBC, ABC,

$$\therefore \{ AP_1, II_1 \} = -1,$$

P_1 being the point where II_1 meets the plane BCD. But

$$\{ XP_1, \alpha\alpha_1 \} = \{ AP_1, II_1 \} = -1.$$

Similarly if Q_1, R_1, S_1 are the points where I_2I_a, I_3I_b, I_4I_c meet the plane BCD, then Q_1, R_1, S_1 are the harmonic conjugates of X with respect to I_2I_a, etc.

It can be proved that P_1, Q_1, R_1, S_1 are the vertices of a quadrangle of which the triangle ABC is the diagonal triangle.

6. The following are some properties of the tetrahedron in which opposite edges are equal :—

(1) Of the eight spheres, only *five* are finite.

(2) Their radii are given by

$$2r = r_1 = r_2 = r_3 = r_4 = \frac{1}{\sqrt{2}} \left\{ \frac{1}{d^2 - a^2} + \frac{1}{d^2 - b^2} + \frac{1}{d^2 - c^3} \right\}^{-\frac{1}{2}} \Big]$$

where d is the diameter of the circumsphere and a, b, c are the edges.

(3) The points α_2, α_3, α_4, β_1, β_3 β_4, γ_1, γ_2, γ_4, δ_1, δ_2, δ_3, all lie on the circumsphere.

(4) $II_1 = II_2 = II_3 = II_4 = R$.

(5) The inscribed spheres of the tetrahedra ABCD, $I_1I_2I_3I_4$ coincide.

C. SALDANHA AND B. B. BAGI.

Solutions.

Question 1257.

(A. C. L. WILKINSON) :—Solve the partial differential equation :

$$r(1-q^2)^2+2pqs(1-q^2)+p^2q^2t-(1-q^2)(rt-s^2)z=0.$$

Solution by Martyn Thomas.

The given equation is of the type $Rr+Ss+Tt+U(rt.-s^2)=V$, where $R=(1-q^2)^2$, $S=2pq(1-q^2)$, $T=p^2q^2$, $U=(q^2-1)z$ and $V=0$, (discussed fully in Forsyth's *Differential Equation.*)

The auxiliary equation for λ is

$$\lambda^2(RT+UV)+\lambda US+U^2=0.$$

$$\therefore\ \lambda^2(1-q^2)^2p^2q^2+\lambda(q^2-1)z.2pq(1-q^2)+(q^2-1)^2z^2=0.$$

$$\therefore\qquad (1-q^2)^2\{\lambda pq-z\}^2=0,$$

giving two equal roots for λ.

$$\therefore\qquad \lambda_1=\lambda_2=\frac{z}{pq}.$$

An intermediary integral is to be sought by solving the system

$$Udy+\lambda_1Tdx+\lambda_1Udp=0, \qquad \dots\quad (1)$$

$$Udx+\lambda_2Rdy+\lambda_2U\,dq=0, \qquad \dots\quad (2)$$

$$dz=pdx+qdy. \qquad \dots\quad (3)$$

Now (1) gives

$$-(1-q^2)zdy+\frac{z}{pq}p^2q^2\,dx+\frac{z^2}{pq}(q^2-1)\,dp=0.$$

$$\therefore\qquad -pq(1-q^2)\,dy+p^2q^2\,dx-z(1-q^2)\,dp=0.$$

$$\therefore\qquad pq^2(pdx+qdy)-pqdy-z(1-q^2)\,dp=0.$$

$$\therefore\qquad pq^2.dz-pq.dy-z(1-q^2)\,dp=0.$$

i.e.,

$$qdz-dy-\frac{z(1-q^2)}{pq}\,dp=0.\ \dots \qquad \dots\quad (4)$$

(2) gives

$$-(1-q^2)z.dx+\frac{z}{pq}(1-q^2)^2\,dy+\frac{z}{pq}(q^2-1)\,z\,dq=0.$$

$\therefore\ -pq\,.dx + (1-q^2)\,dy - zdq = 0;\ 1-q^2 = 0$ giving $q = \pm 1$,

so that $$z = \pm\, y + \phi(x) + \text{const.}$$

$$\therefore\ -q\,(pdx + qdy) + dy - zdq = 0.$$

i.e., $$-qdz + dy - zdq = 0. \quad \ldots \quad \ldots (5)$$

Adding (4) and (5), $$\frac{z(1-q^2)}{pq}\,dp + zdq = 0.$$

$$\therefore\ \frac{qdq}{q^2-1} = \frac{dp}{p}.$$

$$\therefore\ \tfrac{1}{2}\log(q^2-1) = \log p + \text{const.}$$

$$\therefore\ p = k\,(q^2-1)^{\frac{1}{2}} \quad \ldots \quad \ldots (6)$$

$$\therefore\ z = ax + by + c,$$

where $$a = k\,(b^2-1)^{\frac{1}{2}},$$

constants being replaced by functions of $-z \pm y + \phi(x)$.

Question 1263.

(I. TOTADRI) :—Sum the series :

$$1 - \frac{1-x^m}{1-x} + \frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-x^2)} - \frac{(1-x^m)(1-x^{m-1})(1-x^{m-2})}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots\ldots$$

Solution by Hemraj.

Let us write

$$f(x, m) = 1 - \frac{1-x^m}{1-x} + \frac{(1-x^m)(1-x^{m-1})}{(1-x)(1-x^2)} - \ldots\ldots\ldots$$
$$= 1 - (m, 1) + (m, 2) - (m, 3) + \ldots\ldots\ldots$$

Since

$$(m, r) = \frac{[(1-x^{m-r}) + x^{m-r}(1-x^r)]\,(1-x^{m-1})\ldots\ldots\ldots(1-x^{m-r+1})}{(1-x)(1-x^2)\ldots\ldots\ldots(1-x^r)}$$
$$= (m-1, r) + x^{m-r}(m-1, r-1)$$

$$\therefore\ f(x, m) = (1-x^{m-1}) - (1-x^{m-2})(m-1, 1) + (1-x^{m-3})(m-1, 2) - \ldots\ldots\ldots$$
$$= (1-x^{m-1})\,f(x, m-2) = \ldots\ldots\ldots$$
$$= (1-x^{m-1})(1-x^{m-3})\ldots(1-x^{m-2r+1})\,f(x, m-2r)$$

(*a*) If m is a positive odd integer,

$$f(x, m) = 0$$

identically, because the terms equi-distant from the beginning and end cancel each other.

(*b*) If m is a positive even integer,

$$f(x, m) = (1 - x^{m-1})(1 - x^{m-3}) \ldots\ldots (1 - x)\, f(x, 0)$$
$$= (1 - x^{m-1})(1 - x^{m-3}) \ldots\ldots (1 - x).$$

(*c*) If m is a finite number and $|\, x \,| > 1$,

$$f(x, m) = (1 - x^{m-1})(1 - x^{m-3}) \ldots\ldots (1 - x^{m-2r+1})\, f(x, m-2r)$$

where

$$f(x, m-2r) = 1 - \frac{1 - x^{m-2r}}{1 - x} + \frac{(1 - x^{m-2r})(1 - x^{m-2r-1})}{(1 - x)(1 - x^2)} - \ldots$$

$$= 1 + \frac{x^{-1}}{1 - x^{-1}} + \frac{x^{-3}}{(1 - x^{-1})(1 - x^{-2})}$$

$$+ \ldots + \frac{x^{-n(n+1)/2}}{(1 - x^{-1}) \ldots (1 - x^{-n})} + \ldots \text{ when } r \to \infty$$

$$= \prod_{n=1}^{\infty} (1 + x^{-n})$$

It is well-known that if

$$Q_1 = \Pi\,(1 + q^{2n}),$$
$$Q_2 = \Pi\,(1 + q^{2n-1}),$$
$$Q_3 = \Pi\,(1 - q^{2n-1})$$

where $|\, q \,| < 1$, then

$$Q_1 Q_2 Q_3 = 1.$$

Thus
$$\frac{1}{\Pi\,(1 - q^{2n-1})} = \Pi\,(1 + q^n)$$

Hence it follows that

$$f(x, m) = \prod_{r=1}^{\infty} \left[\frac{1 - x^{m-2r+1}}{1 - x^{-2r+1}}\right].$$

(*d*) If $m = -\infty$ and $|\, x \,| > 1$,

$$f(x, m) = 1 + \sum_{n=1}^{\infty} \frac{x^{-n(n+1)/2}}{(1 - x^{-1})(1 - x^{-2}) \ldots\ldots (1 - x^{-n})}$$

$$= \prod_{n=1}^{\infty} (1 + x^{-n})$$

(e) If $|x| \to 1$ in case of each term,

$$f(x, m) = 1 - m + \frac{m(m-1)}{2!} - \frac{m(m-1)(m-2)}{3!} + \ldots\ldots$$

which converges absolutely or diverges according as $m \gtrless 0$.

(Case of the Binomial Theorem.)

(*f*) If $m = +\infty$ and $|x| < 1$,

$$f(x, m) = 1 - \frac{1}{1-x} + \frac{1}{(1-x)(1-x^2)}$$

$$- \frac{1}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$

in which the terms increase and $u_n \to l > 1$.

The series is non-convergent. (*Vide*: Bromwich, Ch. XI).

(*g*) If m is finite and $|x| < 1$,

$$f(x, m) = (1-x^{m-1})(1-x^{m-3})\ldots(1-x^{m-2r+1}).f(x, m-2r)$$

where $$f(x, m-2r) = 1 - \frac{1}{1-x} + \frac{1}{(1-x)(1-x^2)}$$

$$- \frac{1}{(1-x)(1-x^2)(1-x^3)} + \ldots\ldots$$

when $r \to \infty$ which is (*f*).

The other cases need not be considered.

[*Partial solutions by K. Satyanarayana and M. V. Ramakrishnan.*]

Question 1265.

(A. A. KRISHNASWAMI IYENGAR) :—Show that every odd number can be expressed as the sum of seven squares (excepting 1, 3, 5, 9, 11, 17).

Remarks by 'Anon.'

In the solution published at page *147* of the June number (J.I.M.S., Vol. XV, No. 9) the following expression occurs :—" Let p be any prime contained in the form $(3^{2x+1}.n - 2)$." From the summary in Dickson's *History of the Theory of Numbers* (Vol. II, p. 290), it appears that Sylvester meant that the form $(3^{2x+1}.n - 2)$ includes "*composite as well as prime numbers.*" The abovementioned expression must therefore be understood in the sense that p is a prime *of the form* $(3^{2r+1}.n - 2)$, and *equal* to it; not merely a *factor* of it.

The enunciation of Q. 1265 is unnecessarily restricted. Since every positive integer can be expressed as the sum of *four* squares (some of which may be zero), it follows that the number

$$N - \sum_{1}^{k} (\alpha_r)^2 \equiv \text{sum of } four \text{ squares.}$$

$$\therefore \qquad N \equiv \text{sum of } (k+4) \text{ squares,}$$

where the α's can be arbitrarily chosen subject to the obvious limitation that the sum of their squares is less than N. The theorem enunciated in the question is a very special case of the above.

Question 1266.

(A. A. KRISHNASAMY IYENGAR) :—Find general expressions for the sides of rational triangles the squares of whose areas are perfect cubes. (Ex.: 5, 6, 7.)

Solution by I. Totadri Iyengar.

The square of the area of a triangle ABC is

$$\frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{16}.$$

To get the simplest expression, we put

$$\begin{aligned} a+b+c &= 2p^2 \\ a+b-c &= 2q \\ b+c-a &= 2q^2 \\ c+a-b &= 2p. \end{aligned}$$

The relation satisfied by p and q is

$$p^2 = p + q + q^2.$$

or $$p = q + 1.$$

$$\begin{aligned} \therefore \quad a+b+c &= 2(q+1)^2 \\ a+b-c &= 2q \\ b+c-a &= 2q^2 \\ c+a-b &= 2(q+1), \end{aligned}$$

whence $a = 2q+1$, $b = q^2+q$, and $c = q^2+q+1$.

Question 1270.

(R. VAIDYANATHASWAMI) :—If a quadric has double three-point contact with a twisted cubic, show that it touches all the osculating planes of the curve.

Solution by Hemraj.

The parametric equations of a twisted cubic may be written

$$x : y : z : t = \theta^3 : \theta^2 : \theta : \tfrac{1}{3}.$$

The planes $x = 0$, $t = 0$ are the osculating planes at $\theta = 0$ and $\theta = \infty$ respectively.

The quadric $ax^2 + by^2 + cz^2 + dt^2 + 2fyz + 2gzx$

$$+ 2hxy + 2lxt + 2myt + 2nzt = 0$$

meets the cubic at $\theta = 0$ and $\theta = \infty$, each in three consecutive points

if $\qquad a = 0,\ h = 0,\ d = 0,\ n = 0,\ b + 2g = 0,\ c + \dfrac{2m}{3} = 0;$

and the tangent planes to the quadric at $\theta = 0$, $\theta = \infty$ are the osculating planes thereat, if

$$a = 0 \quad h = 0 \quad g = 0 \quad m = 0 \quad n = 0 \quad d = 0.$$

$$\therefore \qquad b = 0 \text{ and } c = 0. \qquad \ldots \qquad \ldots \quad (1)$$

Hence the equation of the quadric having double three-point contact with the cubic at $\theta = 0$, $\theta = \infty$ is

$$2fyz + 2lxt = 0. \qquad \ldots \qquad \ldots \quad (2)$$

The tangent plane at any point $(x'y'z't')$ to the surface is

$$lt'x + fz'y + fy'z + lx't = 0. \ \ldots \qquad \ldots \quad (3)$$

The osculating plane at any point (θ) of the curve is

$$\frac{x}{3} - \theta y + \theta^2 z - \theta^3 t = 0. \qquad \ldots \qquad \ldots \quad (4)$$

(3) and (4) are identical if

$$3lt' = \frac{fz'}{-\theta} = \frac{fy'}{\theta^2} = \frac{lx'}{-\theta^3}$$

i.e., if $\qquad 3l^2x't' = f^2y'z'.$ But $fy'z' + lx't' = 0.$

Hence if $3l + f = 0$, then (2) touches all the osculating planes, as is otherwise obvious.

The tangential equation of the quadric is

$$\text{AX}^2 + \text{BY}^2 + \text{CZ}^2 + \text{DT}^2 + 2\text{FYZ} + 2\text{GZX} + 2\text{HXY}$$
$$+ 2\text{LXT} + 2\text{MYT} + 2\text{NZT} = 0.$$

The co-ordinates of any osculating plane are

$$X : Y : Z : T = \tfrac{1}{3} : -\theta : \theta^2 : -\theta^3.$$

It follows that osculating planes touch the quadric for all values of θ, if

$$A = 0, H = 0, D = 0, N = 0, B + \tfrac{2}{3} G = 0,$$
$$C + 2 M = 0, F + \tfrac{1}{3} L = 0.$$

All these are satisfied except

$$F + \tfrac{1}{3} L = 0$$

when (1) holds.

$$F + \tfrac{1}{3} L = 0$$

is satisfied only if $l + \frac{1}{3}f = 0$.

Hence (2) takes the form $3yz - xt = 0$.

It follows that any quadric having merely double three-point contact with a twisted cubic, cannot be touched by all the osculating planes of the curve.

Now the quadric $3yz = xt$ satisfies all the conditions of the question, any twisted cubic can be transformed linearly into any other and all twisted cubics have the same projective properties. Hence the result.

Question 1271.

(K. Satyanarayana) :—(i) p being any odd prime, prove that

$$2^{p-1} - 1 \not\equiv 0 \pmod{p^2}.$$

Hence or otherwise establish that the numerator of

$$\left[1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \dots\dots \pm \frac{1}{\frac{1}{2}(p-1)}\right] \not\equiv 0 \pmod{p}.$$

Remarks by 'Anon.'

The question is incorrect. It has been known for a long time that the congruence

$$2^{p-1} \equiv 1 \pmod{p^2},\ p \text{ a prime}$$

has a solution $p = 1093.$

To verify this, we have

$$2^{14} = 16384 = -11 + 15\,p.$$

$$\therefore \qquad 2^{28} \equiv 121 - 330\,p \pmod{p^2}$$

and

$$\left.\begin{aligned} 9.2^{28} &\equiv 1089 - 2970\,p \\ &\equiv -4 - 1876\,p \end{aligned}\right\};$$

or

$$9.2^{26} \equiv -(1 + 469\,p).$$

Again,

$$\left.\begin{aligned} 9^7 .\, 2^{26\cdot 7} &\equiv -(1 + 469\,p) \\ &\equiv -(1 + 3283\,p) \\ &\equiv -(1 + 4\,p) \end{aligned}\right\};$$

and

$$\left.\begin{aligned} 9^7 &\equiv (3^7)^2 \equiv (2187)^2 \\ &\equiv (1 + 2p)^2 \equiv 1 + 4\,p. \end{aligned}\right\}$$

Substituting, we get

$$2^{182} \equiv -1 \pmod{p^2}$$

and

$$2^{1092} = 2^{182\cdot 6} \equiv 1;$$

i.e.,

$$(2^{p-1} - 1) \equiv 0 \pmod{p^2}. \text{ for } p = 1093.$$

N.B.—It has been verified (*vide*: any detailed account of Fermat's last Theorem) that this is the only solution less than 2000. No solution greater than 2000 is known. Perhaps some of our readers will find one.

Question 1273.

(S. AUDINARAYANAN) :—Fill up the vacant cells in the following square with integers in G. P. such that the continued product of the numbers along each row, column or diagonal is the same.

1			
			2

Solution by Mrs. Edith Thomas.

The common ratio of the G. P. is evidently 2.

Since the product $2^a . 2^b . 2^c . 2^d, = 2^{a+b+c+d}$ the question reduces to the selection of sets of four numbers from among the group

$$2^0, 2^1, 2^2, 2^3, \ldots\ldots 2^{15},$$

such that the sum of the indices of the four numbers in any set may be the same for all the sets.

Arranging the indices 0, 1, 2, 3, ... 15 in the following manner :—

$$\begin{array}{cccc} 0, & 9, & 6, & 15 \\ 11, & 7, & 8, & 4 \\ 14, & 2, & 13, & 1 \\ 5, & 12, & 3, & 10 \end{array}$$

it is found that the sum along any row, column or diagonal is the same, namely 30.

Hence the corresponding numbers required to fill in the vacant cells of the given square are

$$\begin{array}{cccc} 1, & 2^9, & 2^6, & 2^{15} \\ 2^{11}, & 2^7, & 2^8, & 2^4 \\ 2^{14}, & 2^2, & 2^{13}, & 2 \\ 2^5, & 2^{12}, & 2^3, & 2^{10} \end{array}$$

Thus the continued product in any direction $= 2^{30}$.

Other arrangements of the indices suggested by N. P. Pandya, Hemraj and N. Maitra respectively are :—

0	12	7	11
15	3	8	4
10	6	13	1
5	9	2	14

0	11	7	12
14	5	9	2
13	6	10	1
3	8	4	15

0	3	13	14
8	11	5	6
15	4	10	1
7	12	2	9

Questions for Solution.

1346. (S. D. CHOWLA) :—If $| x | < 1$, prove that

$$\left\{ \frac{(1-x^2)(1-x^4)(1-x^6)(1-x^8)\ldots\ldots}{(1-x)(1-x^3)(1-x^5)(1-x^7)\ldots\ldots} \right\}^8 = \sum_{r=1}^{\infty} \left(\frac{r^3 x^{r-1}}{1-x^{2r}} \right).$$

Deduce that the number of representations of an odd number as the sum of eight squares is 16 times the sum of the cubes of its divisors.

1347. (S. N. KUMARASWAMY) :—ABC is any triangle and P a given point within it. Show how to inscribe a triangle in it having the point P for

(i) centroid, (ii) orthocentre, (iii) incentre, (iv) symmedian point, (v) circumcentre.

1348. (S. D. CHOWLA) :—Prove that

(i) $$\frac{e^{-\pi}}{1+e^{-\pi}} + \frac{3e^{-3\pi}}{1+e^{-3\pi}} + \frac{5e^{-5\pi}}{1+e^{-5\pi}} + \ldots\ldots = \frac{1}{24};$$

(ii) $$\frac{e^{-2\pi}}{1-e^{-2\pi}} + \frac{2^5 e^{-4\pi}}{1-e^{-4\pi}} + \frac{3^5 e^{-6\pi}}{1-e^{-6\pi}} + \ldots\ldots = \frac{1}{504}.$$

1349. (V. TIRUVENKATACHARYA) :—Show that in 'hyperbolic geometry' the volume of a cylinder of radius a and length x is given by

$$V = \pi\, k^2 x \sinh^2\left(\frac{a}{k}\right)$$

where k is the space constant.

1350. (V. A. MAHALINGAM) :—Examine the correctness of the following—

"If $u_1, u_2, u_3, \ldots$ are solutions of

$$\frac{\partial u}{\partial t} = a^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right)$$

then their product $(u_1 u_2 u_3 \ldots)$ is also a solution of the same."

(Whittaker and Watson : *Modern Analysis*).

1351. (S. N. KUMARASWAMY) :—Find three numbers such that their sum is a perfect cube and the sum of their cubes is a perfect fourth power.

1352. (K. J. SANJANA) :—Show how to find integers less than given integer N and having the greatest possible number of divisors.

Example : $N = 150{,}000$.

1353. (ENQUIRER) :—Show that

$$\int_0^\infty \frac{\sin 2nx \sin^m x}{x^{a+1}}\,dx = \frac{\pi\,\Delta^m (2n-m)^a}{2^{m+1}\,\Gamma(a+1)\cos\frac{a+m}{2}\pi},$$

if $2n > m > a$, all being positive, and Δ is the operator in Finite-differences, equivalent to

$(E - 1)$ or $(e^D - 1)$. [Boole's *Finite Differences.*]

1354. (K. J. SANJANA) :—P and Q are points on an ellipse whose eccentric angles are θ and ϕ; P$'$ and Q$'$ are the corresponding points on the nth pedal of the ellipse with respect to the focus S. Prove that

$$\left(\frac{SP}{SP'}\right)^{\frac{2}{n}} + \left(\frac{SQ}{SQ'}\right)^{\frac{2}{n}} = \frac{2 - e^2(\cos^2\theta + \cos^2\phi)}{1 - e^2}.$$

1355. (MARTYN M. THOMAS) :—Show that

$$\int \left[\frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{xdy - ydx}{uv}\right] = 2 \log \frac{v}{u},$$

where $u \equiv ax^2 + 2bxy + by^2$, $v = Ax^2 + 2Hxy + By^2$.

1356. (R. VAIDYANATHASWAMY) :—Prove the following construction for finding the point of contact of the pedal line of a point P with its envelope : Let O be the orthocentre and PQ the chord parallel to the pedal line of P. Produce QP to Q$'$ so that PQ$'$ = PQ. The required point of contact is the intersection of the pedal line with OQ$'$.

Is this construction known ?

1357. (R. VAIDYANATHASWAMY) :—A, B are two points in a plane, so related that when A is fixed, the locus of B is the circle SAA$'$, where A$'$ is the point such that SA = SA$'$ and $A\hat{S}A'$ = a fixed angle θ, S being a fixed point. If B is fixed, what is the locus of A ?

1358. (V. RAMASWAMI AIYAR) :—Four mutually orthocentric points A, B, C, D are the centres of four mutually orthogonal circles. If any point P invert with respect to these circles into P$'$, Q$'$, R$'$, S$'$ then the set [P$'$, Q$'$, R$'$, S$'$] will invert with respect to each of these circles into the same set of points [P, Q, R, S]. If G, G$'$ be the centroids of the sets (P, Q, R, S) and (P$'$, Q$'$, R$'$, S$'$) show that G, G$'$ are inverses of one another with respect to the nine-point circle of the system (A, B, C, D).

Examine what happens to the theorem if P were on one of the circles, say, the circle with centre A.

Notes and Questions.

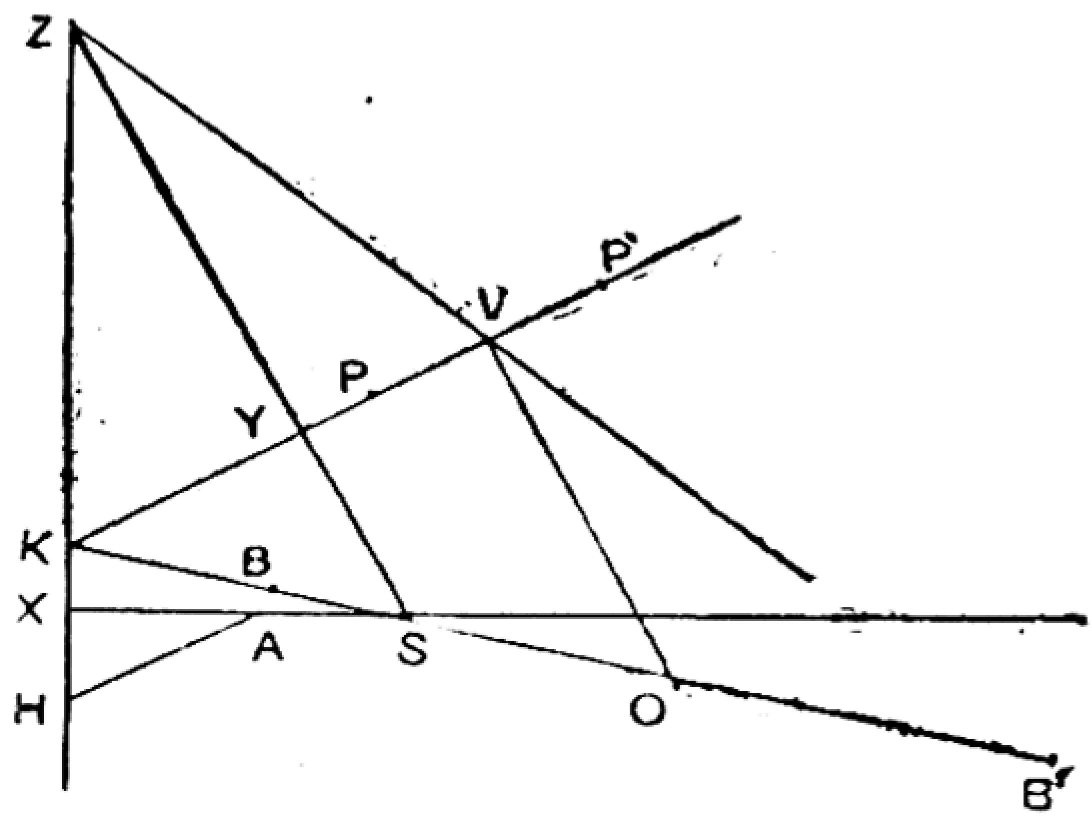
Note on the locus of the middle points of a system of parallel chords of a Conic.

Let S be the focus, A the vertex and X the foot of the corresponding directrix. Draw AH in the direction of the parallel chords to meet the directrix at H, and let KPP′ be any chord of the system cutting the directrix at K.

We know that the points P, P′ on the conic are such that

$$SP : PK = SP' : P'K = SA : AH$$

and therefore if B,B′ divide SK in the ratio SA : AH, the circle on BB′ as diameter passes through P, P′.



Bisect BB′ at O and draw perpendiculars SY, OV on PP′. Produce SY to meet the directrix at Z.

Now $\quad SB : BK = SB' : B'K = SA : AH =$ constant

$\therefore \quad OS : OB = OB : OK =$ constant

$\therefore \quad OS : OK =$ constant.

Hence YV : KV = constant, where V is the middle point of PP′.

Since SZ is drawn in a fixed direction, Z is a fixed point and since K and Y move on fixed lines through Z, V also moves on a fixed line through Z.

In other words, the locus of V is a straight line.

K. SATYANARAYANA.

S. V. RANADE'S PERPETUAL CALENDAR.

Table I.
Guiding Days of the years.

									← B.D./A.D. →															
									←	−201	−101	−1/0	100	200	300									
									400	500	600	700	800	900	1000	Julian								
									1100	1200	1300	1400	1500	1600	1700									
									1500	1600	...	1700	...	1800	...	Gregorian								
									1900	2000	...	2100	——	→	...									
0	...	...	...	...	...	...	...	...	*M*/M	*S*	*St*	*F*/F	*Th*	*W*/W	*Tu*	*								
1	7	*12*	18	...	29	35	*40*	46	Tu	M	S	St	F	Th	W	...	57	63	*68*	74	...	85	91	*96*
2	...	13	19	*24*	30	...	41	47	W	Tu	M	S	St	F	Th	*52*	58	...	69	75	*80*	86	...	97
3	*8*	14	...	25	31	*36*	42	...	Th	W	Tu	M	S	St	F	53	59	*64*	70	...	81	87	*92*	98
...	9	15	*20*	26	...	37	43	*48*	F	Th	W	Tu	M	S	St	54	...	65	71	*76*	82	...	93	99
4	10	...	21	27	*32*	38	...	49	St	F	Th	W	Tu	M	S	55	*60*	66	...	77	83	*88*	94	...
5	11	*16*	22	...	33	39	*44*	50	S	St	F	Th	W	Tu	M	...	61	67	*72*	78	...	89	95	...
6	...	17	23	*28*	34	...	45	51	M	S	St	F	Th	W	Tu	*56*	62	...	73	79	*84*	90	...	...

N.B.—Figure in 'Italics' denotes Leap Year.

The year 4000 A.D. is suggested to be treated as an 'Ordinary year' and not 'Leap'; hence a slight change will be required in the order of Century years thereafter, in Table I.

* Italic letters in half the square denote that only Julian Years were Leap.

Table II.

	JAN.	→(*Leap Year*)→			*FEB.*	
Jan. Oct.	April July	Sept. Dec.	June	Feb. March Nov.	Aug.	May
SLIDING (Week-Day-Tape)						
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31	–	–	–	–

[January and February in Leap years are to be noted in the upper line.]

KEY.

(i) Find out in Table I the 'Guiding Day' of the year in question. The day in the *vertical* line of the 'Century' year and *horizontal* line of the 'Odd' year is the Guiding Day of the year.

(ii) Place this 'Guiding Day' below the *required month* in Table II, [by sliding the Week-Day-Tape, or otherwise.]

And we get the Calendar for the Month.

Examples:—To find the day of the week on

1. 1st January 1925—In Table I the day in the vertical line of 1900 and horizontal line of 25 is Thursday, which is the 'Guiding Day' of the year. Placing it below January, we have the 1st on the same day.

2. 18th June 1815 (Battle of Waterloo)—Sunday is the 'Guiding Day' of the year. Placing it below June, we have the 18th on the same day.

3. 3rd August 1492 (when Columbus set sail for America)—Monday is the 'Guiding Day' of the year. Placing it below August, we see that the 3rd is Friday.

4. 1st January 100 B. C.—In the vertical line of (– 101) and the horizontal line of 1 (– 101 + 1 = – 100), Sunday is the 'Guiding Day.' Placing it below January, we have the 1st on the same day.

The General Equation of the Second Degree in Areals.

Let the equation be

$$S \equiv ux^2 + vy^2 + wz^2 + 2u'yz + 2v'zx + 2w'xy = 0, \ldots \quad (1)$$

and let (α, β, γ) be a focus. The distance d of a point (x, y, z) from the focus is given by

$$2d^2 = a'(x-\alpha)^2 + b'(y-\beta)^2 + c'(z-\gamma)^2 \quad \ldots \quad (2)$$

where a', b', c' stand for $b^2 + c^2 - a^2$, $c^2 + a^2 - b^2$, $a^2 + b^2 - c^2$ respectively, a, b, c being the sides of the triangle of reference.

Now, the equation to the conic can be written in the form

$$d^2 = k(lx + my + nz)^2 \quad \ldots \quad \ldots \quad (1')$$

where $lx + my + nz = 0$ is the directrix corresponding to the focus (α, β, γ). Hence comparing (1) and (1′),

$$-S + 2\lambda d^2 \quad \ldots \quad \ldots \quad (3)$$

can be made a complete square by properly choosing λ, and then its square root represents the directrix of the conic S.

2. Eliminating z from (1) and (2) by means of the relation

$$x + y + z = 1,$$

we get

$$-S \equiv Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0,$$

where, $A = 2v' - w - u$, $B = 2u' - w - v$, $H = u' + v' - w - w'$,

$G = w - v'$, $\quad F = w - u'$, $\quad C = -w$,

and $2d^2 = x^2(a' + c') + 2xyc' + y^2(b' + c') - 2x\{a'\alpha + c'(\alpha + \beta)\}$

$$-2y\{b'\beta + c'(\alpha + \beta)\} + a'\alpha^2 + b'\beta^2 + c'(\alpha + \beta)^2.$$

If the second degree terms in (3), namely,

$$x^2\{A + \lambda(a' + c')\} + 2xy(H + \lambda c') + y^2\{B + \lambda(b' + c')\} \quad (4)$$

form a square, then

$$\{\lambda(a' + c') + A\}\{(\lambda(b' + c') + B\} = (H + \lambda c')^2,$$

and on inserting the values of A, B, H, we have

$$\lambda^2(b'c' + c'a' + a'b') + \lambda[a'(2u' - v - w) + b'(2v' - w - u) + c'(2w' - u - v)]$$
$$+ (vw + wu + uv - 2uu' - 2vv' - 2ww')$$
$$+ (2v'w' + 2w'u' + 2u'v' - u'^2 - v'^2 - w'^2) = 0. \quad \ldots \quad (5)$$

For the value of λ given by (5), the expression (4) takes the form

$$\frac{1}{m}\,[mx + ny]^2,$$

where $m = \mathrm{A} + \lambda\,(a' + c')$ and $n = \mathrm{H} + \lambda\, c'$. The directrix is therefore of the form $mx + ny + p = 0$,

so that $$-\mathrm{S} + 2\lambda\, d^2 = \frac{1}{m}\,[mx + ny + p]^2. \quad \ldots \quad (6)$$

On equating the co-efficients of x, y and the absolute terms on both sides of (6), we have,

$$\lambda\alpha\,(a' + c') + \lambda\beta c' = \mathrm{G} - p, \ \ldots \quad \ldots \quad (7)$$

$$\lambda\alpha c' + \lambda\beta\,(b' + c') = \mathrm{F} - \frac{np}{m}, \quad \ldots \quad (8)$$

and $$\lambda(\alpha + \beta)^2 c' + \lambda a'\alpha^2 + \lambda b'\beta^2 = \frac{p^2}{m} - \mathrm{C};$$

which may be reduced by (7) and (8) to

$$\alpha\,(\mathrm{G} - p) + \beta\left(\mathrm{F} - \frac{np}{m}\right) = \frac{p^2}{m} - \mathrm{C}. \ \ldots \quad \ldots \quad (9)$$

Eliminating α, β from (7), (8) and (9), we get

$$\begin{vmatrix} \lambda(a' + c') & \lambda c' & \mathrm{G} - p \\ \lambda c' & \lambda(b' + c') & \mathrm{F} - \frac{np}{m} \\ \mathrm{G} - p, & \mathrm{F} - \frac{np}{m}. & \frac{p^2}{m} - \mathrm{C} \end{vmatrix} = 0 \quad \ldots \ (10)$$

which determines p; and α, β are then given by (7) and (8).

If p is eliminated from (7) and (8), we obtain the equation to the axis on which the focus (α, β) lies; it is

$$x\,\{a'n + c'(n - m)\} + y\,\{c'(n - m) - b'm\}$$
$$+ (x + y + z)\left[\frac{(v' - w)n}{\lambda} - \frac{(u' - w)m}{\lambda}\right] = 0. \quad \ldots \quad (11)$$

The eccentricity of the conic may now be deduced. It turns out to be

$$\frac{1}{2\Delta\sqrt{2\lambda}}\left[a^2(\lambda a' + v' + w' - u - u') + c^2(\lambda c' + u' + v' - w - w')\right.$$
$$\left. + \frac{b^2 n}{m}(u + u' - v' - w' - \lambda a')\right]^{\frac{1}{2}} \quad \ldots \quad (12)$$

where Δ is the area of the triangle of reference.

3. If the conic S is a circle, then the directrix

$$mx + ny + p = 0$$

should be at infinity. The directrix in homogeneous form is

$$mx + ny + p(x + y + z) = 0$$

i.e.,
$$x(m + p) + y(n + p) + pz = 0; \qquad \ldots \quad (13)$$

and this should be identical with the line at infinity, *viz.*,

$$x + y + z = 0$$

and hence,
$$m + p = n + p = p,$$

giving
$$m = 0,\ n = 0;$$

hence we obtain the well-known conditions

$$\lambda = \frac{v + w - 2u'}{2a^2} = \frac{w + u - 2v'}{2b^2} = \frac{u + v - 2w'}{2c^2}.$$

Also from (6), S may be written in the form

$$\frac{1}{m}\,[mx + ny + p]^2 - 2\lambda d^2 = 0. \qquad \ldots \quad (14)$$

When S is a circle, d is equal to the radius, and on making m and n tend to zero in (14), we get

$$\lim.\ p^2/m = 2\lambda d^2, \quad \ldots \qquad \ldots \quad (15)$$

and the determinant of (10) becomes

$$\begin{vmatrix} a' + c' & c' & G \\ c' & b' + c' & F \\ G & F & \lambda\,(2\lambda d^2 - C) \end{vmatrix} = 0;$$

or,

$$2\lambda^2 d^2 (b'c' + c'a' + a'b') + \begin{vmatrix} a' + c' & c' & w - v' \\ c' & b' + c' & w - u' \\ w - v' & w - u' & w\lambda \end{vmatrix} = 0, \qquad (16)$$

which determines the radius of the circle, and

$$b'c' + c'a' + a'b' = 16\Delta^2.$$

The centre is given by

$$\alpha\,(a' + c') + \beta c' = \frac{w - v'}{\lambda},\quad \alpha c' + \beta\,(b' + c') = \frac{w - u'}{\lambda}.$$

K. D. PANDAY, M.A., B.Sc.

Solutions.

Question 1274.

(A. A. KRISHNASWAMI IYENGAR) :—Prove the following approximate construction for the rectification of a circle:—

Draw AB, CD two perpendicular diameters. Along AO take $AK = \frac{1}{8}$ AO. Bisect CO at E. Join CK and from it cut off CG = CO and through G draw GF parallel to KE to meet CE at F. Draw a circle to pass through G and F so as to touch CD at F. Let this circle cut CG again at H.

Then the circumference of the circle ADBC is 6OC+2CH.

Solution by S. A. Mani, Hemraj, P. R. Venkatakrishna Iyer K. Satyanarayana, M. V. Ramakrishnan, etc.

Proof : Since CF touches the circle FGH,

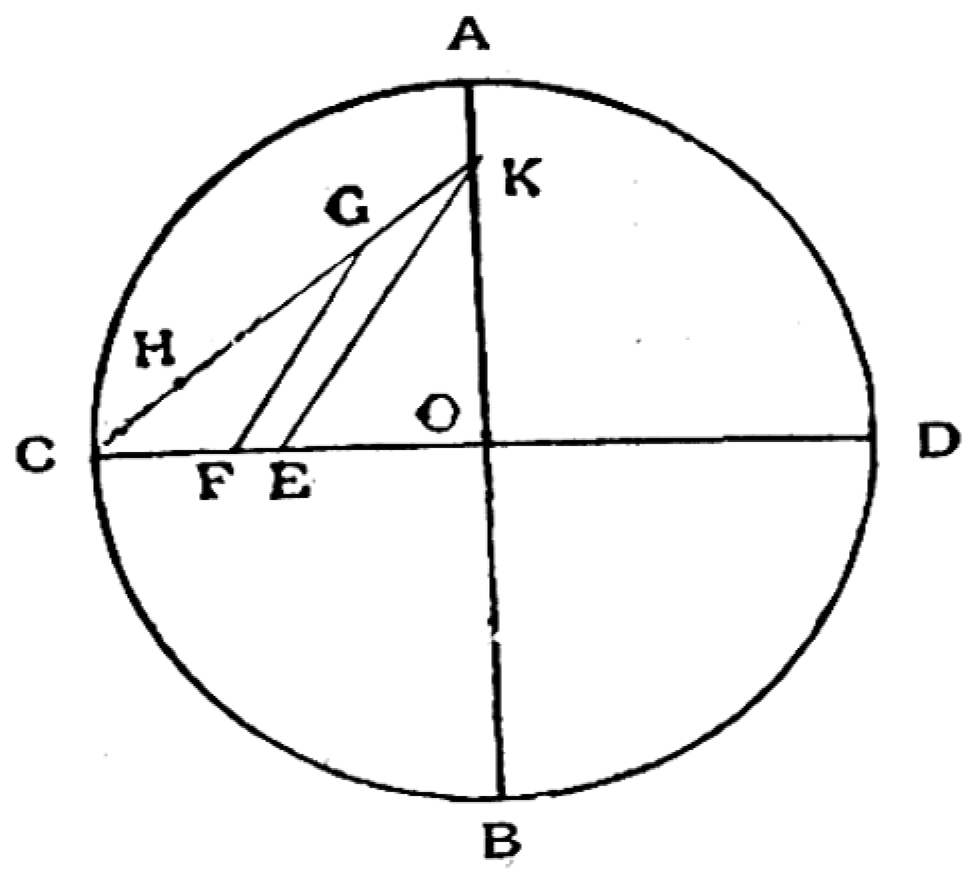
$$CF^2 = CH.CG = CH.r$$

where r = radius.

$$\therefore \qquad CH = \frac{CF^2}{r}. \qquad \dots \qquad \dots \quad (1)$$

Also
$$\frac{CF}{CE} = \frac{CG}{CK}.$$

$$\therefore \qquad CF = \frac{CG.CE}{CK} = \frac{r^2}{2CK}. \qquad \dots \qquad \dots \quad (2)$$



Again $$CK^2 = r^2\left[1+\left(\frac{7}{8}\right)^2\right] = \frac{113}{64}r^2. \quad \ldots \quad (3)$$

$$\therefore \quad CH = \frac{CF^2}{r} = \frac{1}{r}\cdot\frac{r^4}{4CK^2} = \frac{r^3}{4}\cdot\frac{64}{113r^2} = \frac{16r}{113}.$$

$$\therefore \quad 6OC+2CH = 6r + \frac{32r}{113}$$

$$= 2r\,(3\cdot 1415929\ldots\ldots)$$

which agrees with the value $2\pi r$ as far as the 6th decimal place.

Question 1276.

(G. V. TELANG):—If x represents the area of a triangle and x_1, x_2 those of the pedal triangles of the positive (or negative) Brocard point and the 3rd Brocard point, show that

$$xx_2 = 2x_1\,(x - 2x_1).$$

Solution by Hemraj.

Brocard's first triangle $A'B'C'$ is in perspective in three ways with the original triangle ABC:

AC', BA', CB' intersect at Ω (*first* or positive Brocard point)

AB', BC', CA' „ at Ω' (*second* or negative Brocard point)

AA', BB', CC' „ at Ω'' (*third* Brocard point).

The trilinear co-ordinates of Ω, Ω', Ω'' are

$$\left(2R\sin^2 w\,\frac{c}{b},\ 2R\sin^2 w\,\frac{a}{c},\ 2R\sin^2 w\,\frac{b}{a}\right)$$

$$\left(2R\sin^2 w\,\frac{b}{c},\ 2R\sin^2 w\,\frac{c}{a},\ 2R\sin^2 w\,\frac{a}{b}\right)$$

$$\left(2R\sin^2 w\,\frac{bc}{a^2},\ 2R\sin^2 w\,\frac{ca}{b^2},\ 2R\sin^2 w\,\frac{ab}{c^2}\right)$$

where w is the Brocard angle of the triangle ABC.

Also $\Omega A = 2R\sin w\,\frac{b}{a}$, $\Omega B = 2R\sin w\,\frac{c}{b}$, $\Omega C = 2R\sin w\,\frac{a}{c}$.

Now $2x = \Sigma\, \Omega B\,.\,\Omega C \sin C = \dfrac{2R\sin^2 w}{abc}(a^2b^2 + b^2c^2 + c^2a^2)$;

$$2x_1 = \frac{2R\sin^4 w}{abc}(a^2b^2 + b^2c^2 + c^2a^2).$$

(The pedal triangles of Ω and Ω' are equal in area.)

$$2x_2 = \frac{2R\sin^4 w}{abc}(a^4 + b^4 + c^4).$$

Since $$2\cos 2w = \frac{a^4 + b^4 + c^4}{a^2b^2 + b^2c^2 + c^2a^2},$$

$\therefore$ $$xx_2 = 2x_1(x - 2x_1).$$

Question 1277.

(S. RAJANARAYANAN) :—ABC is a triangle; (D, E), (F, G) are points in AB, AC, such that AD = BE = AF = CG. If DG, FE meet BC in H, K, prove that BH = CK.

Solution by Martyn M. Thomas, M.D. Bhat, B.B. Bagi, M. V. Ramakrishnan, Hemraj, V. V. S. N. Murthy, S. Audinarayanan, K. Satyanarayana, S. A. Mani, P. R. Venkatakrishna Aiyar, N. P. Pandya, C. Ranganathan, etc.

Since DGH and FEK are transversals to the $\triangle$ ABC,

$$\frac{BH}{HC}\cdot\frac{CG}{GA}\cdot\frac{AD}{DB} = 1 = \frac{CK}{KB}\cdot\frac{BE}{EA}\cdot\frac{AF}{FC}.$$

Cancelling common factors, $\dfrac{BH}{HC} = \dfrac{CK}{KB}.$

$\therefore$ $$\frac{BH}{BH - HC} = \frac{CK}{CK - KB}.$$

$\therefore$ $$\frac{BH}{BC} = \frac{CK}{BC}.$$

$\therefore$ $$BH = CK.$$

Question 1280.

(M. BHIMASENA RAO and the late C. KRISHNAMACHARI) :—If

$$S_r = 1 + \frac{1}{2^r} + \frac{1}{3^r} \ldots\ldots,$$

and γ is Euler's constant, show that

$$\int_0^\infty e^{-x} (\log x)^3 \, dx = -(\gamma^3 + 3\gamma S_2 + 2S_3);$$

$$\int_0^\infty e^{-x} (\log x)^4 \, dx = \gamma^4 + 6\gamma^2 S_2 + 8\gamma S_3 + 3S_2^2 + 6S_4;$$

and show how to evaluate generally $\int_0^\infty e^{-x} (\log x)^n \, dx$.

Solution by Martyn Thomas.

Now $$\int_0^\infty e^{-x} \,.\, x^{n-1} \, dx = \Gamma(n).$$

Differentiating successively with respect to n,

$$\int_0^\infty e^{-x} x^{n-1} \log x \, dx = \Gamma'(n),$$

$$\int_0^\infty e^{-x} x^{n-1} (\log x)^2 \, dx = \Gamma''(n),$$

$$\int_0^\infty e^{-x} x^{n-1} (\log x)^3 \, dx = \Gamma'''(n),$$

Generally, $$\int_0^\infty e^{-x} x^{n-1} (\log x)^r \, dx = \Gamma^r(n).$$

Putting $n = 1$, we obtain

$$\int_0^\infty e^{-x} (\log x)^3 \, dx = \Gamma''' \;\; (1),$$

....

and generally, $$\int_0^\infty e^{-x} (\log x)^r \, dx = \Gamma^r \;\; (1).$$

Now

$$\frac{\Gamma'(n)}{\Gamma(n)} + \gamma = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots\ldots + \frac{1}{n-1} = \sum_{k=0}^{\infty} \left(\frac{1}{1+k} - \frac{1}{n+k}\right) \quad \ldots (1)$$

[*Vide*: Wilson's *Calculus*, § 149-(17), or Bromwich's *Infinite Series.*]

Differentiating successively with respect to n,

$$\frac{\Gamma''(n)}{\Gamma(n)} - \left\{\frac{\Gamma'(n)}{\Gamma(n)}\right\}^2 = \sum_{k=0}^{\infty} \frac{1}{(n+k)^2} \quad \ldots \quad (2)$$

$$\frac{\Gamma'''(n)}{\Gamma(n)} - 3\frac{\Gamma''(n)\,\Gamma'(n)}{[\Gamma(n)]^2} + 2\left\{\frac{\Gamma'(n)}{\Gamma(n)}\right\}^3 = \sum_{k=0}^{\infty} \frac{-2}{(n+k)^3} \quad \ldots \quad (3)$$

$$\frac{\Gamma^{iv}(n)}{\Gamma(n)} - 4\frac{\Gamma'''(n)\,\Gamma'(n)}{[\Gamma(a)]^2} + 12\frac{\Gamma''(n)\,[\Gamma'(n)]^2}{[\Gamma(n)]^3}$$

$$-3\left\{\frac{\Gamma''(n)}{\Gamma(n)}\right\}^2 - 6\left\{\frac{\Gamma'(n)}{\Gamma(n)}\right\}^4 = \sum_{k=0}^{\infty} \frac{6}{(n+k)^4}. \quad \ldots \quad (4)$$

Putting $n = 1$ in (1), (2), (3), (4) and remembering $\Gamma(1) = 1$, we get

$$\Gamma'(1) = -\gamma \text{ and } \Gamma''(1) = \gamma^2 + S_2$$

$$\Gamma'''(1) = -(\gamma^3 + 3\gamma S_2 + 2S_3)$$

$$\Gamma^{iv}(1) = \gamma^4 + 6\gamma^2 S_2 + 8\gamma S_3 + 3S_2^2 + 6S_4.$$

Hence the required results.

Question 1281.

(M. Bhimasena Rao and the late C. Krishnamachari):—

If $$S_n = \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)\frac{1}{n},$$

show that

$$\lim_{n \to \infty} \left\{ (S_1 + S_2 + \ldots + S_n) - \gamma \log n - \tfrac{1}{2}(\log n)^2 \right\} = \frac{\gamma^2}{2} + \frac{\pi^2}{12}.$$

Further show that

$$S_1 + S_2 + \ldots\ldots + S_n = \binom{n}{1} - \binom{n}{2}\left(\frac{1}{2^2}\right) + \binom{n}{3}\frac{1}{3^2}$$

$$- \ldots\ldots + (-1)^{n-1}\binom{n}{n}\frac{1}{n^2},$$

where $\binom{n}{r}$ = number of combinations of n things r at a time.

Solution by Martyn Thomas.

First Part.

$$S_1 = \frac{1}{1^2}$$

$$S_2 = \left(1 + \frac{1}{2}\right)\frac{1}{2} = 1 \cdot \frac{1}{2} + \frac{1}{2^2}$$

$$S_3 = 1 \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{3^2}$$

$$S_4 = 1 \cdot \frac{1}{4} + \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{4^2}$$

....

$$S_n = 1 \cdot \frac{1}{n} + \frac{1}{2} \cdot \frac{1}{n} + \frac{1}{3} \cdot \frac{1}{n} + \ldots\ldots + \frac{1}{(n-1)n} + \frac{1}{n^2}$$

$$\therefore \quad \sum_1^n S_r = \sum_1^n \left(\frac{1}{r^2}\right) + \left\{ \sum_2^n \left(\frac{1}{r}\right) \right.$$

$$\left. + \frac{1}{2} \sum_3^n \left(\frac{1}{r}\right) + \ldots\ldots + \left(\frac{1}{n-1} \cdot \frac{1}{n}\right) \right\}$$

$$\text{Now } \gamma = \lim_{n \to \infty} \left[\sum_1^n \left(\frac{1}{r}\right) - \log n \right]$$

$$\therefore \quad \gamma^2 = \lim_{x \to \infty} \left[\left(\sum_1^n \left(\frac{1}{r}\right) \right)^2 \right.$$

$$\left. - 2 \log n \left(\sum_1^n \left(\frac{1}{r}\right) \right) + (\log n)^2 \right]$$

$$= \lim_{n \to \infty} \left[\left(\sum_1^n \frac{1}{r^2} \right) - 2 \log n \, (\gamma + \log n) + (\log n)^2 \right.$$

$$\left. + 2 \left\{ 1 \cdot \frac{1}{2} + \ldots\ldots + \frac{1}{2} \cdot \frac{1}{3} + \ldots\ldots + \frac{1}{n-1} \cdot \frac{1}{n} \right\} \right]$$

$$= \lim_{n \to \infty} \left[\frac{\pi^2}{6} + 2 \left\{ \sum_1^n S_r - \frac{\pi^2}{6} \right\} - 2\gamma \log n - (\log n)^2 \right]$$

$$\therefore \quad \frac{\gamma^2}{2} + \frac{\pi^2}{12} = \lim_{n \to \infty} \left[\sum_1^n S_r - \gamma \log n - \frac{1}{2} (\log n)^2 \right].$$

Second Part.

Consider the fraction

$$\frac{1}{(x+1)(x+2)\ldots\ldots(x+n)},$$

which, when decomposed into its partial fractions

$$\frac{A_1}{x+1}+\frac{A_2}{x+2}+\ldots\ldots+\frac{A_n}{x+n},$$

will have

$$A_1=\frac{1}{1.2\ldots\ldots(n-1)};\ A_2=\frac{1}{(-1))(1.2\ldots\ldots(n-2)};$$

$$A_3=\frac{1}{(-2)(-1)1.2\ldots\ldots(n-3)};$$

$$A_{n-1}=\frac{1}{(-n+2)(-n+3)\ldots\ldots(-2)(-1)1},$$

$$A_n=\frac{1}{(-n+1)(-n+2)\ldots\ldots(-2)(-1)}.$$

$$\therefore \frac{1}{n!\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\ldots\ldots\left(1+\frac{x}{n}\right)}=\frac{1}{(n-1)!}\cdot\frac{1}{1+x}$$

$$-\frac{1}{(n-2)!}\cdot\frac{1}{2\left(1+\frac{x}{2}\right)}+\ldots\ldots+\frac{(-1)^{n-1}}{(n-1)!}\,\frac{1}{n\left(1+\frac{x}{n}\right)}.$$

Multiply both sides by $(n-1)!$; then

$$\frac{1}{n\left(1+\frac{x}{1}\right)\left(1+\frac{x}{2}\right)\cdots\left(1+\frac{x}{n}\right)}=(1+x)^{-1}-\frac{n-1}{1!}\cdot\frac{1}{2}\left(1+\frac{x}{2}\right)^{-1}+$$

$$\ldots\ldots+(-1)^{n-1}\cdot\frac{1}{n}\left(1+\frac{x}{n}\right)^{-1}.$$

$$\therefore\quad \frac{1}{n}\left\{1+x\left(1+\frac{1}{2}+\ldots+\frac{1}{n}\right)+x^2\left(\frac{1}{1.2}+\frac{1}{1.3}+\ldots\right)+\ldots\right\}^{-1}$$

$$=\text{right side.}$$

Equate the co-efficient of x on both sides and change sign.

$$\therefore S_n\equiv\frac{1}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\ldots+\frac{1}{n}\right)$$

$$=1-\binom{n-1}{1}\cdot\frac{1}{2^2}+\binom{n-1}{2}\cdot\frac{1}{3^2}-\ldots+(-1)^{n-2}\binom{n-1}{n-1}\cdot\frac{1}{n^2}.$$

By the Method of Finite Differences

$$\sum_1^n S_r = \Delta^{-1}\,[S_{n+1}]$$

$$= \Delta^{-1}\left[1 - \binom{n}{1}\cdot\frac{1}{2^2} + \binom{n}{2}\frac{1}{3^2} + \ldots\ldots(-1)_n\binom{n}{n}\frac{1}{(n+1)^2}\right]$$

But $\qquad {}_nC_r + {}_nC_{r-1} = {}_{n+1}C_r$

$\therefore \qquad \Delta\,[{}_nC_r] = {}_nC_{r-1}$

$\therefore \qquad \Delta^{-1}\,[{}_nC_{r-1}] = {}_nC_r$

$\therefore$

$$\sum_1^n S_r = n - \binom{n}{2}\frac{1}{2^2} + \binom{n}{3}\frac{1}{3^2} - \ldots\ldots + (-1)^{n-1}\binom{n}{n}\cdot\frac{1}{n^2},$$

since $\Delta^{-1}\binom{n}{n} = {}_nC_{n+1} = 0.$

Question 1287.

(B. B. Bagi):—If regular polygons $A_1A_2 \ldots\ldots A_m$; $B_1B_2 \ldots\ldots B_n$ are inscribed in circles with centres a_1, a_2 and radii r_1, r_2 respectively, then show that

$$\sum_{x=1}^{m}\sum_{y=1}^{n}(A_xB_y)^2 = mn\,[(a_1a_2)^2 + r_1^2 + r_2^2].$$

[The circles are not necessarily in one plane.]

Solution (1) *by Martyn Thomas* (2) *by K. J. Sanjana.*

(1) Lemma. If D, E, F ...... are fixed points in space, G their centre of mean position, and P any variable point, then

$$\Sigma\,PD^2 = \Sigma\,GD^2 + n\,.\,PG^2.$$

Choosing P to coincide with $A_1, A_2 \ldots A_m$ successively, and D, E, F ... being respectively placed at the angular points $B_1, B_2, B_3 \ldots\ldots B_n$, and remembering that G must coincide with the centre a_2, of the circle of radius r_2, we have

$$\sum_{y=1}^{n}(A_1B_y)^2 = \sum_{y=1}^{n}(a_2B_y)^2 + n\,(A_1a_2)^2, \text{ by Lemma}$$

$$= nr_2^2 + n\,(A_1a_2)^2$$

Similarly, $\sum_{y=1}^{n} (A_2B_y)^2 = n \cdot r_2^2 + n(A_2a_2)^2$

$$\sum_{y=1}^{n} (A_3B_y)^2 = nr_2^2 + n(A_3a_2)^2$$

....

Finally, $\sum_{y=1}^{n} (A_mB_y)^2 = n \cdot r_2^2 + n(A_ma_2)^2$

Summing up these 'm' equalities,

$$\sum_{x=1}^{m} \sum_{y=1}^{n} (A_xB_y)^2 = m(n\, r_2^2) + n \sum_{x=1}^{m} (a_2A_x)^2$$

$$= mn\, r_2^2 + n\,[\sum_{x=1}^{m} (a_1A_x)^2 + m(a_1a_2)^2],$$

by Lemma

$$= mn(r_2^2 + n\,[m\, r_1^2 + m\,(a_1a_2)^2]$$
$$= mn\,[r_2^2 + r_1^2 + (a_1a_2)^2].$$

(2) Let A_xM be the $\perp$r from A_x on the plane of the circle $B_1B_2 \ldots\ldots B_n$; let the line B_1a_2 make with Ma_2 an angle θ. Join A_xB_y, MB_y.

Then $A_xB_y^2 = A_xM^2 + MB_y^2 = A_xM^2 + Ma_2^2 + a_2B_y^2$

$$- 2\, Ma_2 \cdot a_2B_y \cdot \cos\left(\theta + \frac{2y\pi}{n}\right)$$

$$= A_xa_2^2 + r_2^2 - 2r_2 . Ma_2 \cos\left(\theta + \frac{2y\pi}{n}\right).$$

$$\therefore \sum_{y=1}^{y=n} A_xB_y^2 = n\, A_xa_2^2 + nr_2^2 - 2r_2 . Ma_2 . \sum_{1}^{n} \cos\left(\theta + \frac{2y\pi}{n}\right)$$

$$= n\, A_xa_2^2 + nr_2^2.$$

$$\therefore \sum_{x=1}^{x=m} \sum_{y=1}^{y=n} A_xB_y^2 = n \sum_{1}^{m} A_xa_2^2 + nmr_2^2$$

$$= n(m \cdot a_1a_2^2 + mr_1^2) + nmr_2^2,$$

by similar reasoning,

$$= nm(a_1a_2^2 + r_1^2 + r_2^2).$$

[*Additional Analytical Solutions by Nandlal M. Mehta and I. Totadri Iyengar.*]

Question 1291.

(MARTYN M. THOMAS, M.A.):—If the equation of a curve in multiple angular co-ordinates be

$$\theta_1 + \theta_2 + \theta_3 + \ldots\ldots + \theta_n = \text{a constant},$$

show that the equation of its orthogonal trajectory in multiple polar co-ordinates is

$$r_1\, r_2\, r_3 \ldots\ldots r_n = \text{a constant}.$$

[*Particular case*—The orthogonal trajectories of rectangular hyperbolas are Cassini's Ovals.]

Solution (1) *by A. Narasinga Rao,* (2) *by Miss Y. Bhate, I. B. Mukherji, M. V. Seshadri, S. Audinarayanan, S. M. Shah, C. Ranganatha Iyengar, M. K. Kewalramani and K. N. Srikanta Sastry.*

(1) It is a well-known result that, if $u(x, y) + iv(x, y)$ where u and v are real, is a function of the complex variable $x + iy$, then the curves $u =$ const. and $v =$ const. are orthogonal.

Consider the function

$$\phi(z) \equiv \log(z - \alpha_1) + \log(z - \alpha_2) + \ldots\ldots + \log(z - \alpha_n)$$

where $\alpha_1 \alpha_2 \ldots\ldots \alpha_n$ are complex numbers corresponding to the n poles $O_1\, O_2 \ldots\ldots O_n$.

The real part of $\phi(z)$ is $\Sigma \log r$, while the unreal part is $\Sigma\, i\theta$; on equating these to arbitrary constants, we obtain the two series of curves mentioned in the question.

(2) The differential equation of the curve is

$$\sum_{t=1}^{n} \frac{d\theta_t}{ds} = 0$$

where ds is an element of the arc of the curve. If ϕ_t denote the angle made by the tangent at any point with the radius vector r_t, the equation may be written

$$\sum_{t=1}^{n} \frac{\sin \phi_t}{r_t} = 0.$$

The orthogonal trajectory is obtained by changing every ϕ_t into

$$\left(\phi_t + \frac{\pi}{2}\right).$$

Hence its differential equation is

$$\sum_{t=1}^{n} \frac{\cos\phi_t}{r_t} = 0, \text{ i.e., } \sum_{t=1}^{n} \frac{1}{r_t}\frac{dr_t}{ds} = 0, \text{ i.e., } \sum_{t=1}^{n} \frac{dr_t}{r_t} = 0.$$

Integrating $\sum_{t=1}^{n} \log r_t \equiv \log r_1 r_2 \ldots\ldots r_n$ = a constant.

$$\therefore \qquad r_1 r_2 \ldots\ldots r_n = \text{a constant.}$$

[When $n = 2$, $\theta_1 + \theta_2 = k$. This represents a rectangular hyperbola, the two poles being the extremities of the major axis. The orthogonal trajectories are $r_1 r_2$ = a constant and are Cassini's Ovals.]

Question 1292.

(V. Ramaswamy Aiyar) :—If a and b be positive and unequal, show that

(1) $$\frac{\frac{1}{2}(a^2 - b^2)}{a - b} < \frac{\frac{1}{3}(a^3 - b^3)}{\frac{1}{2}(a^2 - b^2)} < \frac{\frac{1}{4}(a^4 - b^4)}{\frac{1}{3}(a^3 - b^3)} < \ldots\ldots\ldots$$

without limit.

(2) Show further that

$$\left\{\frac{a^m - b^m}{m(a - b)}\right\}^{\frac{1}{m-1}}$$

always increases as m increases.

(3) Hence or otherwise, show that

$$\frac{a^m - b^m}{a - b} \gtrless m\,(ab)^{\frac{m-1}{2}}, \text{ according as } m\,(m^2 - 1) \gtrless 0,$$

and that

$$\frac{a^m - b^m}{a - b} \gtrless m\left(\frac{a + b}{2}\right)^{m-1}, \text{ according as } m\,(m - 1)\,(m - 2) \gtrless 0.$$

(4) Show also that

$$\sqrt{(ab)} < \frac{a - b}{\log a - \log b} < \frac{a^{\frac{a}{a-b}} \cdot b^{\frac{b}{b-a}}}{e} < \frac{a + b}{2}.$$

Solution and Remarks by J. B. Freeman, M. A., L. T.

§ 1. Let $u_{m-1} \equiv \dfrac{\frac{1}{m}(a^m - b^m)}{\frac{1}{m-1}(a^{m-1} - b^{m-1})}$.

Then $u_m > u_{m-1}$

if $$\frac{1}{m^2-1}(a^{m-1} - b^{m-1})(a^{m+1} - b^{m+1}) > \frac{1}{m^2}(a^m - b^m)^2,$$

i.e., $$\frac{1}{m^2-1}[a^{2m} - a^m b^m \left(\frac{a}{b} + \frac{b}{a}\right) + b^{2m}]$$

$$> \frac{1}{m^2}[a^{2m} - 2a^m b^m + b^{2m}]$$

or $$m^2 . [a^{2m} - a^m b^m \left(\frac{a^2+b^2}{ab}\right) + b^{2m}]$$

$$> (m^2 - 1)[a^{2m} - 2a^m b^m + b^{2m}]$$

or $$-m^2 . a^m b^m \left(\frac{a^2+b^2}{ab}\right) > -a^{2m} + 2a^m b^m - b^{2m} - 2m^2 a^m b^m,$$

or $$m^2 a^m b^m \left(\frac{a^2+b^2}{ab} - 2\right) < (a^m - b^m)^2,$$

i.e., if $$m^2 . a^{m-1} b^{m-1} (a-b)^2 < (a^m - b^m)^2$$

or $$\frac{1}{m} . \frac{a^m - b^m}{a-b} > a^{\frac{m-1}{2}} . b^{\frac{m-1}{2}}.$$

$$\text{L. H. S.} = \frac{a^{m-1} + a^{m-2}b + a^{m-3}b^2 + \ldots\ldots + ab^{m-2} + b^{m-1}}{m}$$

$$> \sqrt[m]{\{a^{(m-1)+(m-2)+\ldots\ldots+2+1} \times b^{(m-1)+(m-2)+\ldots\ldots+2+1}\}}$$

i.e., $$> \sqrt[m]{\left\{a^{\frac{m(m-1)}{2}} . b^{\frac{m(m-1)}{2}}\right\}}$$

i.e., $$> a^{\frac{m-1}{2}} b^{\frac{m-1}{2}}.$$

§ 1.1. In the classic equality

$$\frac{f(a)-f(b)}{\mathrm{F}(a)-\mathrm{F}(b)}=\frac{f'(\xi)}{\mathrm{F}'(\xi)},\ (b<\xi<a),$$

if we put $$f(x) = x^m \text{ and } \mathrm{F}(x) = x^{m-1},$$

we get $$\frac{a^m-b^m}{a^{m-1}-b^{m-1}}=\frac{m\,\xi^{m-1}}{(m-1)\,\xi^{m-2}}$$

or $$u_{m-1}\equiv\frac{(m-1)(a^m-b^m)}{m(a^{m-1}-b^{m-1})}=\xi.$$

Hence ξ is an increasing function of m.

§ 2. If u_m has the value assigned above, we find that

$$\left[\prod_{r=1}^{m-1}u_r\right]^{\frac{1}{m-1}}=\left\{\frac{1}{m}\,\frac{a^m-b^m}{a-b}\right\}^{\frac{1}{m-1}}$$

and $$\left[\prod_{r=1}^{m}u_r\right]^{\frac{1}{m}}=\left\{\frac{1}{m+1}\,\frac{a^{m+1}-b^{m+1}}{a-b}\right\}^{\frac{1}{m}}.$$

Thus $$\left[\prod_{r=1}^{m}u_r\right]^{\frac{1}{m}}>\left[\prod_{r=1}^{m-1}u_r\right]^{\frac{1}{m-1}},$$

if $$u_1^{m-1}.u_2^{m-1}\ldots\ldots u_m^{m-1}>u_1^m.u_2^m\ldots\ldots u_{m-1}^m$$

that is, if $$u_m^{m-1}>u_1u_2\ldots\ldots u_{m-1}$$

which is true as $$u_m>u_{m-1}>u_{m-2}\text{ etc.};$$

Hence the required result follows.

§ 3. Applying the result obtained in § 1.1, we have

$$\left\{\frac{a^m-b^m}{m(a-b)}\right\}^{\frac{1}{m-1}}>\left\{\frac{a^2-b^2}{2(a-b)}\right\}^{\frac{1}{2-1}},\text{ if } m\geqslant 2;$$

i.e., $$>\frac{a+b}{2}.$$

$$>\sqrt{ab}.$$

$$\therefore\quad\frac{a^m-b^m}{a-b}>m\,(ab)^{\frac{m-1}{2}}.$$

See also § 1.

If $m<-1$, put $m=-n$ and apply the result where $n>1$, and a and b are unequal as before.

§ 3.1. We have shown that

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}}$$

is an increasing function of m.

Hence if $m > 2$

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}} > \left\{\frac{a^2 - b^2}{2(a-b)}\right\}^{\frac{1}{2-1}}$$

i. e.,
$$> \frac{a+b}{2}.$$

$$\therefore \qquad \left\{\frac{a^m - b^m}{m(a-b)}\right\} > \left\{\frac{a+b}{2}\right\}^{m-1}$$

provided that $m > 2$.

If $m = 2$ or 1, the inequality becomes an equality.

N.B.—From this result by putting $b = ak$ it follows that

$$\frac{1 + k + k^2 + \ldots\ldots + k^{m-1}}{m} > \left\{\frac{1+k}{2}\right\}^{m-1},$$

if $m > 2$ and $k > 0$.

§ 3.2. All the above inequalities have been proved for the case where m is a positive integer.

In the case where m is fractional of the form p/q, we have by writing

$$A^q = a, \quad B^q = b,$$

$$\left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}} = \left\{\frac{a^{\frac{p}{q}} - b^{\frac{p}{q}}}{\frac{p}{q}(a-b)}\right\}^{\frac{1}{\frac{p}{q}-1}} = \left\{\frac{\frac{1}{p}(A^p - B^q)}{\frac{1}{q}(A^q - B^p)}\right\}^{\frac{1}{\frac{p}{q}-1}}$$

$$= u_{p+1} \cdot u_p \cdot u_{p-1} \ldots\ldots u_{p-q}, \text{ if } p > q$$

$$= \prod_{s=0}^{s=p}(u_{s+1}) \Big/ \prod_{s=0}^{s=q}(u_{s+1})$$

where u has the same meaning as in § 1, and p and q have integral values.

This is an increasing function of p when q is fixed, or a decreasing function of q when p is fixed.

Hence the result (2) is true when m is fractional.

When m is negative, let $m = -n$ where n is positive; then write

$$v_m \equiv \left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}} = \left[\frac{a^{-n} - b^{-n}}{-n(a-b)}\right]^{\frac{1}{-n-1}}$$

$$= \left\{\frac{A^n - B^n}{\frac{n}{AB}(A-B)}\right\}^{\frac{1}{-n-1}}, \text{ where } a^{-1} = A,\ b^{-1} = B$$

$$= \left\{\frac{n(A-B)}{AB(A^n - B^n)}\right\}^{\frac{1}{n+1}}$$

$$= \left[\frac{1}{AB}\right]^{\frac{1}{n+1}} \left[\frac{1}{v_n}\right]^{\frac{n-1}{n+1}}$$

which is evidently a decreasing function of n since v_n is an increasing function of n.

Further it is seen that v_m has the limits

$$\frac{a^{\frac{a}{a-b}} \cdot b^{\frac{b}{b-a}}}{e} \cdot \frac{a-b}{\log a - \log b}$$

as m tends to the values 1 and 0, respectively.

§ 3.3. We have shown that

$$\left\{\frac{a^m - b^m}{m\,(a-b)}\right\}^{\frac{1}{m-1}}$$

is a continuous function of m and increases as m increases. ... (2)

§ 3.4. Consider now the function

$$\left\{\frac{a^m - b^m}{(a-b)}\right\} - m\,(ab)^{\frac{m-1}{2}} \equiv f(m)$$

which is a continuous function of m.

This is positive when $m = 2$, as can be seen by applying (2). It is equal 0 when $m = 1$, and when $m = 0$ and also when $m = -1$. Hence between the values -1 and 0, 0 and $+1$, the expression has the same sign. That is, $f(m)$ is positive for $m > 1$, negative for $0 < m < 1$, positive for $-1 < m < 0$ and negative for $m < -1$.

Thus $\dfrac{a^m - b^m}{a - b} \lesseqgtr m\,(ab)^{\frac{m-1}{2}}$, according as $m\,(m^2 - 1) \gtrless 0$.

§ 3.5. Consider the expression

$$\frac{a^m - b^m}{a - b} - m\left(\frac{a+b}{2}\right)^{m-1} \equiv F(m)$$

which is a continuous function of m.

This is positive when $m > 2$ (see § 3), zero when $m = 2$, 1 and 0.

Hence the expression has the same sign for $m > 2$, for $1 < m < 2$, for $0 < m < 1$ and $m < 0$; that is, $F(m)$ is positive for $m < 2$, negative for $1 < m < 2$, positive for $0 < m < 1$ and negative for $m < 0$.

Thus
$$\frac{a^m - b^m}{a - b} \gtrless m\left(\frac{a+b}{2}\right)^{m-1},$$

according as
$$m\,(m-1)\,(m-2) \lesseqgtr 0.$$

§ 4. Let
$$v_m \equiv \left\{\frac{a^m - b^m}{m(a-b)}\right\}^{\frac{1}{m-1}}.$$

Then
$$\sqrt{ab} < \lim_{m\to 0} v_m < \lim_{m\to 1} v'_m < \lim_{m\to 2} v_2$$

or
$$\sqrt{ab} < \frac{a - b}{\log a - \log b} < \frac{a^{\frac{a}{a-b}} \times b^{\frac{b}{b-a}}}{e} < \frac{a+b}{2}.$$

Partial solutions by Hans R. Gupta, A. Mahadevan, and K. N. Srikanta Sastry.

Questions for Solution.

1359. (M. BHIMASENA RAO):—Show that the sixteen points of contact of the in- and ex-circles with the nine-point circle and the sides of a triangle lie on a bi-circular quartic whose focal conics are concentric with the circum-circle of the triangle.

1360. (M. BHIMASENA RAO):—Show that the director circles of the in-conics of a triangle passing through the in-centre (or an ex-centre) and the corresponding Gergonne point touch the in-circle (or the ex-circle).

1361. (M. BHIMASENA RAO):—Evaluate the determinants

$$A \equiv \begin{vmatrix} -\left(a^2+\frac{bcd}{a}\right), & ab+cd, & ac+bd, & ad+bc \\ ba+cd, & -\left(b^2+\frac{acd}{b}\right), & bc+ad, & bd+ac \\ ca+bd, & cb+ad, & -\left(c^2+\frac{abd}{c}\right), & cd+ab \\ da+bc, & db+ac, & dc+ab, & -\left(d^2+\frac{abc}{d}\right). \end{vmatrix}$$

$$B \equiv \begin{vmatrix} -2, & \frac{c}{d}+\frac{d}{c}, & \frac{d}{b}+\frac{b}{d}, & \frac{b}{c}+\frac{c}{b} \\ \frac{c}{d}+\frac{d}{c}, & -2, & \frac{d}{a}+\frac{a}{d}, & \frac{a}{c}+\frac{c}{a} \\ \frac{d}{b}+\frac{b}{d}, & \frac{d}{a}+\frac{a}{d}, & -2, & \frac{a}{b}+\frac{b}{a} \\ \frac{b}{c}+\frac{c}{b}, & \frac{a}{c}+\frac{c}{a}, & \frac{a}{b}+\frac{b}{a}, & -2 \end{vmatrix}$$

and show that $A = B\,(abcd)^2$.

1362. (S. NARAYANA AIYAR):—Show that

$$\frac{1-dn\,u\,dn\,v\,dn\,(u+v)}{1-cn\,u\,cn\,v\,cn\,(u+v)} = k^2.$$

1363. (S. NARAYANA AIYAR) :—In Dr. Glaisher's notation show that

$$(1)\ \ \mathrm{cs}\frac{k}{n}\,.\,\mathrm{cs}\frac{2k}{n}\,.\,\mathrm{cs}\frac{3k}{n}\,\ldots\ldots\,\mathrm{cs}\frac{(n-1)k}{n}=(1-k^2)^{\frac{n-1}{4}}$$

$$(2)\ \ \mathrm{ds}\frac{k+ik'}{n}\,.\,\mathrm{ds}\frac{2}{n}(k+ik')\,.\,\mathrm{ds}\frac{3}{n}(k+ik')\,\ldots\ldots\,\mathrm{ds}\frac{n-1}{n}(k+ik')$$

$$=k^{n-1}\,.\,\{\mathrm{cn}\,(k+ik')\}^{\frac{n-1}{2}}$$

1364. (R. GOPALASWAMI) :—Prove that a conic touching the sides of a triangle ABC at L, M and N has double contact with the polar conic of the centre of perspective of the triangles ABC, LMN; and that the chord of contact is their axis of perspective.

Deduce that the Brocard ellipse has double contact with the circumcircle.

1365. (P.L. SRIVASTAVA):—Solve by the principle of *virtual work* the following :—

" A smooth rod passes through a smooth ring at the focus of an ellipse, whose major axis is horizontal and rests with its lower end on the quadrant of the curve furthest removed from the focus. Show that its length must at least be $\frac{3a}{4}+\frac{a}{4}\sqrt{1+8e^2}$, where a is the semi-major axis and e the eccentricity of the ellipse." [*Math. Tripos*, 1883.]

1366. (S. D. CHOWLA) :—With the usual notation for the Elliptic Functions prove that

$$(\mathrm{i})\ \ \int_0^{\frac{\pi}{2}}\left\{\frac{1-2q\cos 2x+q^2}{1+2q\cos 2x+q^2}\right\}\left\{\frac{1-2q^3\cos 2x+q^6}{1+2q^3\cos 2x+q^6}\right\}$$

$$\times\left\{\frac{1-2q^5\cos 2x+q^{10}}{1+2q^5\cos 2x+q^{10}}\right\}\ldots\ldots dx=\frac{\pi^2}{4\,k\,k'^{\frac{1}{2}}}.$$

$$(\mathrm{ii})\ \ \int_0^{\pi}\sin^3 x\sin px\prod_{n=0}^{\infty}\left\{\frac{1-2q^{2n}\cos 2x+q^{4n}}{1-2q^{2n-1}\cos 2x+q^{4n-2}}\right\}^3 dx=0\,,$$

where p is an even positive integer. Evaluate the definite integral when p is an odd positive integer. ($q=e^{k-\pi'/k}$).

Notes and Questions.

The Six Co-ordinates of a Right Line.

Introduction. The equations to a line passing through the point (a, b, c) and having for its direction cosines (l, m, n) are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}; \qquad \ldots \quad \text{(i)}$$

whence it follows that

$$\left.\begin{aligned} ny - mz &= nb - mc = \lambda, \\ lz - nx &= lc - na = \mu, \\ \text{and} \quad mx - ly &= ma - lb = \nu. \end{aligned}\right\} \qquad \ldots \quad \text{(ii)}$$

The quantities λ, μ, ν are constant for any point on the line. Hence the six constants of a line are $(l, m, n, \lambda, \mu, \nu)$. These are termed "line co-ordinates." Also equations (i) can be written in the form

$$\left.\begin{aligned} x &= Az + B \\ \text{and} \quad y &= Cz + D \end{aligned}\right\},$$

from which it is evident that the equations to a line involve *four* and only four *independent* constants.

Hence we must have two identical relations among the six constants of a line. These are

$$l^2 + m^2 + n^2 = 1 \text{ and } l\lambda + \mu m + n\nu = 0.$$

It may be noted that if p denotes the perpendicular from the origin on the line and (L, M, N) are the direction cosines of the normal to the plane passing through the origin and the line, then

$$\lambda = pL, \ \mu = pM, \ \nu = pN.$$

If the line passes through the origin, it is obvious from (ii), that $\lambda = \mu = \nu = 0$. Take (x, y, z) any point not necessarily on the line and let

$$\begin{aligned} A &\equiv ny - mz - \lambda, \\ B &\equiv lz - nx - \mu, \\ \text{and} \quad C &\equiv mx - ly - \nu. \end{aligned}$$

Then for any point on the line

$$A = 0, B = 0 \text{ and } C = 0.$$

Hence $A = 0$, $B = 0$ and $C = 0$ represent planes through the line parallel to OX, OY and OZ respectively.

In fact, the line $(l, m, n, \lambda, \mu, \nu)$ may be defined as the intersection of any two of the planes $A = 0$. $B = 0$, $C = 0$.

Let $$D \equiv \lambda x + \mu y + \nu z.$$
Then $$Ax + By + Cz + D \equiv 0$$
for any point on the line $A = B = C = 0$.

Hence $D = 0$, or $\lambda x + \mu y + \nu z = 0$ is the equation to the plane passing through the line and the origin.

Take any fixed point (x_1, y_1, z_1) and write

$$A_1 \text{ for } ny_1 - mz_1 - \lambda,$$
$$B_1 \text{ for } lz_1 - nx_1 - \mu,$$
and $$C_1 \text{ for } mx_1 - ly_1 - \nu.$$
Also, let $$D_1 \equiv \lambda x_1 + \mu y_1 + \nu z_1.$$

Now consider $A_1 x + B_1 y + C_1 z + D_1 = 0$.
This represents a plane through $(x_1\, y_1\, z_1)$; for

$$A_1 x_1 + B_1 y_1 + C_1 z_1 + D_1 \equiv 0.$$

Also $A_1 x + B_1 y + C_1 z + D_1 \equiv x_1 A + y_1 B + z_1 C + D$.
Further $$Ax_1 + By_1 + Cz_1 + D = 0$$
is satisfied by any point on the line.

Hence the equation $A_1 x + B_1 y + C_1 z + D_1 = 0$ represents a plane through the point $(x_1\, y_1\, z_1)$ and the line $(l, m, n, \lambda, \mu, \nu)$.

§ 1. *To find the condition that two given lines* $(l, m, n, \lambda, \mu, \nu)$ *and* $(l', m', n', \lambda', \mu', \nu')$ *are coplanar.*

If possible let them meet in (x_1, y_1, z_1). Then

$$\left.\begin{aligned} ny_1 - mz_1 - \lambda &= 0, \\ lz_1 - nx_1 - \mu &= 0, \\ mx_1 - ly_1 - \nu &= 0; \end{aligned}\right\} \text{ and } \left.\begin{aligned} n'y_1 - m'z_1 - \lambda' &= 0, \\ l'z_1 - n'x_1 - \mu' &= 0, \\ m'x_1 - l'y_1 - \nu' &= 0. \end{aligned}\right\}$$

Out of these six equations if $x_1\, y_1\, z_1$ be eliminated, three conditions among the constants must result, out of which two are

$$l\lambda + m\mu + n\nu = 0, \text{ and } l'\lambda' + \mu'm' + n'\nu' = 0. \quad \dots \quad (1)$$

Hence one more condition is necessary for the intersection of the two lines.

Multiply the first set of equations by l', m', n' in order and the second set by l, m, n in order and add; then we get

$$l'\lambda + m'\mu + n'\nu + l\lambda' + m\mu' + n\nu' = 0. \qquad \ldots \quad (2)$$

Supposing this condition is satisfied it is easy to write out the equation to the plane containing the given lines. If the direction cosines of the normal to this plane are a, b, c, then

$$al + bm + cn = 0,$$
$$al' + bm' + cn' = 0.$$

$$\therefore \qquad \frac{a}{mn' - m'n} = \frac{b}{nl' - n'l} = \frac{c}{lm' - l'm}.$$

Also to determine the plane completely, we must know a fixed point through which it passes, and the direction cosines of its normal.

This point can be chosen on any one of the given lines, say the point where the first line meets the plane $x = 0$, *viz.*, $\left(0, \frac{-\nu}{l}, \frac{\mu}{l}\right)$.

The equation of the plane is
$$\begin{vmatrix} x & y + \frac{\nu}{l}, & z - \frac{\mu}{l} \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

That is,

$$x(mn' - m'n) + y(nl' - n'l) + z(lm' - l'm)$$
$$+ \left(\frac{\nu l'n}{l} - \nu n' - \mu m' + \frac{\mu ml'}{l}\right) = 0.$$

The constant term, by the help of (2), becomes

$$\frac{\nu l'n}{l} + \frac{m\mu l'}{l} + l'\lambda + l\lambda' + \mu'm + n\nu'$$
$$= l\lambda' + m\mu' + n\nu', \qquad \text{by (1)}$$
$$= -(l'\lambda + m'\mu + n'\nu).$$

Hence the equation to the plane becomes

$$x(mn' - m'n) + y(nl' - n'l) + z(lm' - l'm)$$
$$+ (l\lambda' + m\mu' + n\nu') = 0,$$

or
$$x(mn' - m'n) + y(nl' - n'l) + z(lm' - l'm)$$
$$-(l'\lambda + m'\mu + n'\nu) = 0.$$

§ 2. *To find the perpendicular distance from $(x_1\, y_1\, z_1)$ to the line $(l, m, n, \lambda, \mu, \nu)$.*

Let the perpendicular meet the line in $(x_2\, y_2\, z_2)$. Then

$$\left.\begin{aligned} ny_2 - mz_2 - \lambda &= 0 \\ lz_2 - nx_2 - \mu &= 0 \\ mx_2 - ly_2 - \nu &= 0. \end{aligned}\right\}$$

Also let

$$\left.\begin{aligned} A_1 &\equiv ny_1 - mz_1 - \lambda \\ B_1 &\equiv lz_1 - nx_1 - \mu \\ C_1 &\equiv mx_1 - ly_1 - \nu. \end{aligned}\right\}$$

Then
$$\begin{aligned} A_1 &= (ny_1 - mz_1 - \lambda) - (ny_2 - mz_2 - \lambda) \\ &= n(y_1 - y_2) - m(z_1 - z_2). \end{aligned}$$
Similarly
$$\begin{aligned} B_1 &= l(z_1 - z_2) - n(x_1 - x_2), \\ C_1 &= m(x_1 - x_2) - l(y_1 - y_2). \end{aligned}$$

Now square and add, then
$$A_1^2 + B_1^2 + C_1^2 = \Sigma(x_1 - x_2)^2.$$
But
$$p^2 = \Sigma(x_1 - x_2)^2.$$
$\therefore$
$$\begin{aligned} p^2 &= A_1^2 + B_1^2 + C_1^2 \\ &= \Sigma(ny_1 - mz_1 - \lambda)^2. \end{aligned}$$

If we replace $(x_1\, y_1\, z_1)$ by current co-ordinates, we find the equation of a right circular cylinder of radius p, of which the given line is the axis. Hence the equation to the cylinder will be
$$p^2 = \Sigma(ny - mz - \lambda)^2.$$

Corollary. To find the equation of a cone of semi-vertical angle α, having the given line as axis and a point O $(x_1\, y_1\, z_1)$ on it as its vertex: Take P $(x\, y\, z)$ any point on the cone, then the square of the length of the perpendicular from $(x\, y\, z)$ on the line is $\Sigma(ny - mz - \lambda)^2$, and this must be equal to $OP^2 \tan^2 \alpha$.

Hence $$[\Sigma(x - x_1)^2] \tan^2 \alpha = \Sigma(ny - mz - \lambda)^2,$$
is the equation of the cone required.

§ 3. *To find the shortest distance between two non-intersecting lines, $(l_1, m_1, n_1, \lambda_1, \mu_1, \nu_1)$ and $(l_2, m_2, n_2, \lambda_2, \mu_2, \nu_2)$.*

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the feet of the common perpendicular, and let (l, m, n) be its direction cosines.

Then $$l\,l_1 + m\,m_1 + n\,n_1 = 0,$$
and $$l\,l_2 + m\,m_2 + n\,n_2 = 0;$$
so that
$$\frac{l}{m_1 n_2 - m_2 n_1} = \frac{m}{n_1 l_2 - n_2 l_1} = \frac{n}{l_1 m_2 - l_2 m_1}$$
$$= \frac{1}{\sqrt{\Sigma (m_1 n_2 - m_2 n_1)^2}} = \frac{1}{\sin\theta}, \quad \ldots \quad (1)$$

where θ is the angle between the two straight lines. Also if p be the shortest distance between the two lines

$$\frac{x_1 - x_2}{p} = l, \quad \frac{y_1 - y_2}{p} = m, \quad \frac{z_1 - z_2}{p} = n. \quad \ldots \quad (2)$$

Again,
$$\left.\begin{array}{l} n_1 y_1 - m_1 z_1 - \lambda_1 = 0 \\ l_1 z_1 - n_1 x_1 - \mu_1 = 0 \\ m_1 x_1 - l_1 y_1 - \nu_1 = 0 \end{array}\right\} \qquad \left.\begin{array}{l} n_2 y_2 - m_2 z_2 - \lambda_2 = 0 \\ l_2 z_2 - n_2 x_2 - \mu_2 = 0 \\ m_2 x_2 - l_2 y_2 - \nu_2 = 0 \end{array}\right\} \quad (3)$$

Multiply the first set by l_2, m_2, n_2 in order and the second set by l_1, m_1, n_1 in order and add, then

$$\Sigma (m_1 n_2 - m_2 n_1)(x_1 - x_2) = l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1;$$

or, using (1) and (2), we get

$$l \sin\theta \,.\, pl + m \sin\theta \,.\, p.m + n \sin\theta \,.\, p.n = l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1;$$

$$\therefore \quad p\,(l^2 + m^2 + n^2) \sin\theta = l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1.$$

$$\therefore \quad p = \frac{l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1}{\sin\theta}.$$

Cor. 1. From this result we can at once determine the condition for the intersection of two lines. For, if the two lines intersect, $p = 0$,

$$\therefore \quad l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1 = 0.$$

Cor. 2. If, however, the two lines are parallel to each other, they intersect each other at infinity and hence

$$l_1 \lambda_2 + m_1 \mu_2 + n_1 \nu_2 + l_2 \lambda_1 + m_2 \mu_1 + n_2 \nu_1 = 0.$$

Also θ, the angle between the lines, is zero. Hence the above method to determine the shortest distance between the lines fails; we investigate p for such lines in the next article.

§ 4. *To find the shortest distance between two parallel lines* $(l, m, n, \lambda_1, \mu_1, \nu_1), (l, m, n, \lambda_2, \mu_2, \nu_2)$.

Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be the feet of any line perpendicular to both, one on each.

Then $$p^2 = \Sigma (x_1 - x_2)^2.$$

Also $$ny_1 - mz_1 - \lambda_1 = 0,$$

and $$ny_2 - mz_2 - \lambda_2 = 0.$$

$$\therefore \quad n(y_1 - y_2) - m(z_1 - z_2) = \lambda_1 - \lambda_2.$$

Similarly $$l(z_1 - z_2) - n(x_1 - x_2) = \mu_1 - \mu_2.$$

and $$m(x_1 - x_2) - l(y_1 - y_2) = \nu_1 - \nu_2.$$

Also $$l(x_1 - x_2) + m(y_1 - y_2) + n(z_1 - z_2) = 0,$$

because the line joining $(x_1 y_1 z_1)$ and $(x_2 y_2 z_2)$ is perpendicular to each of the given lines.

Squaring and adding these four equations, we get

$$\Sigma (x_1 - x_2)^2 = (\lambda_1 - \lambda_2)^2 + (\mu_1 - \mu_2)^2 + (\nu_1 - \nu_2)^2$$

i.e., $$p^2 = \Sigma (\lambda_1 - \lambda_2)^2.$$

§ 5. *To find the condition that three straight lines may be concurrent.*

Of course the three conditions of intersection of the lines taken in pairs must first be satisfied. Then, supposing the lines meet at $(x'\ y'\ z')$, we have also

then
$$\lambda_1 x' + \mu_1 y' + \nu_1 z' = 0$$
$$\lambda_2 x' + \mu_2 y' + \nu_2 z' = 0$$
$$\lambda_3 x' + \mu_3 y' + \nu_3 z' = 0.$$

For the equation $\lambda x + \mu y + \nu z = 0$ is satisfied by the co-ordinates of any point on the line.

Eliminating $(x'\ y'\ z')$, we get the required condition in the form

$$\begin{vmatrix} \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \\ \lambda_3 & \mu_3 & \nu_3 \end{vmatrix} = 0.$$

P. L. SRIVASTAVA.

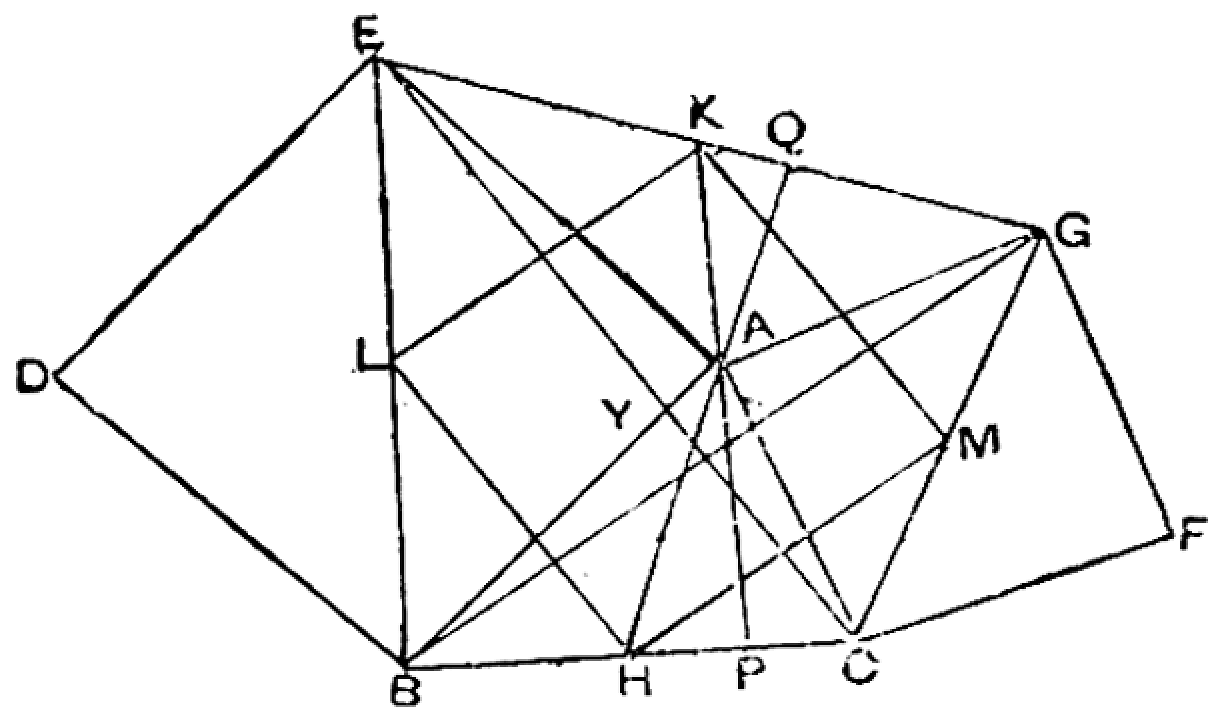
Solutions.

Question 1286.

(B. B. BAGI) :—Squares ABDE and ACFG are described externally on the sides AB, AC of a triangle ABC, L, M being their respective centres. H, K are the mid-points of BC, EG and AK, AH meet them in P, Q ; then show that (1) L, H, P, M, Q, K lie on a circle of which HK LM are perpendicular diameters and that (2) four times the area of square HMKL = sq. AF + sq. AD + 4 Δ ABC.

Solution by I. B. Mukherjee, K. J. Sanjana, S. Audinarayanan, S. N. Kumaraswamy, V. A. Apte, K. R. Srinivasa Iyengar, and R. Srinivasan.

(1) The triangles EAC and BAG being congruent, EC and BG are equal and cut at right angles, for they are equally inclined in the same sense to EA and BA which form a right angle. L, H, M, K being the middle-points of the sides of EBCG form a parallelogram whose adjacent sides LH, HM are parallel and equal to $\frac{1}{2}$ EC and $\frac{1}{2}$ BG respectively. The figure LHMK is therefore a square.



If GA is produced to N such that AN = GA, the triangles EAN and BAC are congruent ; angle ACB = ENA = KAG (since EN is parallel to KA) = complement of PAC. Therefore APC is a right angle, as also AQG for a similar reason. Hence L, H, P, M, Q, K are concyclic having the diagonals of the square LM as perpendicular diameters.

(2) Again area of LHMK $= \frac{1}{2}$ EBCG.

$$\therefore \quad 4\,(\text{square LM}) = 2\,(\text{ABE} + \text{ACG} + \text{ABC} + \text{AEG})$$
$$= \text{sq. AD} + \text{sq. AF} + 4\,\text{ABC},$$

since ABC and AEG are equal in area, having two sides of the one equal .o two sides of the other and the included angles supplementary.

Question 1295.

(M. BHIMASENA RAO) :— Solve,

$$(1 + x^2)^2 \frac{d^2y}{dx^2} + 4x(1 + x^2)\frac{dy}{dx} + (1 + 2x^2)\, y = 0.$$

Solution (1) *by A. Mahalingam, K. N. Srikantha Sastry and M. V. Seshadri*; (2) *by Miss Y. Bhate, Hans R. Gupta, M. K. Kewalramani, I. B. Mukherjee, S. Audinarayanan, A. Mahadevan, J. B. Freeman, I. Totadri Iyengar and S. M. Shah.*

(2) Changing the independent variable by the substitution $x = \tan\theta$, the equation reduces to

$$\frac{d^2y}{d\theta^2} + 2\tan\theta\,\frac{dy}{d\theta} + (1 + 2\tan^2\theta)\, y = 0.$$

This again, on putting $y = ve^{-\int \tan\theta\, d\theta}$, reduces to

$$\frac{d^2v}{d\theta^2} = 0.$$

$\therefore \quad y = (\text{A} + \text{B}\theta)\cos\theta$, A and B being constants of integration.

i.e.,
$$y = \frac{\text{A} + \text{B}\tan^{-1} x}{\sqrt{1 + x^2}}.$$

(2) The differential equation

$$\frac{d^2y}{dx^2} + \frac{4x}{1 + x^2}\frac{dy}{dx} + \frac{1 + 2x^2}{(1 + x^2)^2}\, y = 0$$

when reduced to its normal form becomes

$$\frac{d^2v}{dx^2} - \frac{1}{(1 + x^2)^2}\, v = 0.$$

Then following the particular method given in Forsyth's *Differential Equations*, § 68, p. 118, the differential equation

$$\frac{d\text{P}_1}{dx} - \text{P}_1{}^2 = -\frac{1}{(1 + x^2)^2}$$

has to be solved. Substitute

$$P_1 = \frac{a + bx}{1 + x^2}$$

for trial. If this be a solution the following three equations must be satisfied:

$$\text{(i)} \quad a^2 - b = 1,$$
$$\text{(ii)} \quad b(b + 1) = 0,$$
$$\text{(iii)} \quad 2a(b + 1) = 0;$$

$a = 0$, $b = -1$ satisfy these equations simultaneously.

$$\therefore \qquad P_1 = -\frac{x}{1 + x^2}.$$

Hence $$\frac{dz}{dx} + 2P_1 z = A,$$

where $$z = ve^{-\int P_1 dx} = v\sqrt{1 + x^2},$$

and A is an arbitrary constant.

This equation is linear in z. Integrating it,

$$z = (1 + x^2)\left\{\int \frac{A dx}{1 + x^2} + B\right\}$$
$$= (1 + x^2)\left\{A \tan^{-1} x + B\right\}$$

where B is another arbitrary constant.

Now $$z = v\sqrt{1 + x^2} = ye^{\int \frac{2x}{1+x^2} dx} . \sqrt{1 + x^2} = y(1 + x^2)^{\frac{3}{2}}$$

The solution of the given equation is thus

$$y = \frac{1}{\sqrt{1 + x^2}}\left\{A \tan^{-1} x + B\right\}.$$

Remarks by Mr. J. B. Freeman.

The equation can be written as

$$[(1 + x^2) D + x] \; [(1 + x^2) D + x] \; y = 0$$

and where D stands for the operator $\frac{d}{dx}$, the solution follows easily.

Question 1296.

(V. RAMASWAMI IYER) :—Prove that for every acute-angled triangle ABC, there is an obtuse-angled triangle A′B′C′ and *vice versa*, so that, if O, O′ be their respective ortho-centres

$$OA = O'A', \; OB = O'B', \; OC = O'C'.$$

If the two triangles be so placed that their circum-centres coincide at a point S, then the circum-circle of either will touch the nine-point circle of the other.

Solution (1) *by R. Gopalaswamy,* (2) *by Ram Behari.*

(1) If R and R$'$ be the circum-radii of the two triangles then

$$|R'\cos A'| = R\cos A,\ |R'\cos B'| = R\cos B,$$
$$|R'\cos C'| = R\cos C\ ;$$

that is, the absolute values of the cosines of the angles in the triangle ABC are proportionally increased. If B$'$ is obtuse,

$$\frac{\cos A'}{\cos A} = -\frac{\cos B'}{\cos B} = \frac{\cos C'}{\cos C} = \frac{R}{R'} = k, \text{ say.}$$

Now

$$\cos^2 A' + \cos^2 B' + \cos^2 C' + 2\cos A'\cos B'\cos C' - 1 = 0. \quad (1)$$
$$\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A\cos B\cos C - 1 = 0. \quad (2)$$

From (1)

$$k^2(\cos^2 A + \cos^2 B + \cos^2 C) - 2k^3\cos A\cos B\cos C - 1 = 0.$$

Using (2) we get

$$k^2(-1 + 2\cos A\cos B\cos C) + 2k^3\cos A\cos B\cos C + 1 = 0;$$

or

$$2k^2\cos A\cos B\cos C - k + 1 = 0.$$

Hence
$$k = \frac{1 \pm \sqrt{1 - 8\cos A\cos B\cos C}}{4\cos A\cos B\cos C}.$$

But
$$SO^2 = R^2 - 8R^2\cos A\cos B\cos C;$$

i.e.,
$$k = \frac{2R(R \pm SO)}{R^2 - SO^2}; \text{ or, } k = \frac{2R}{R \mp SO}.$$

Let us for the present confine ourselves to the root

$$k = \frac{2R}{R + SO}.$$

Then
$$R' = R/k = \tfrac{1}{2}(R + SO). \quad \ldots \quad \ldots \quad (1)$$

Hence if we describe a circle with R$'$ as radius and inscribe in it a triangle with angles A$'$, B$'$, C$'$, so that

$$\cos A' = k\cos A,\ \cos B' = -k\cos B,\ \cos C' = k\cos C,$$

we get the triangle required.

To determine SO$'$, we note that if we took our original equations as

$$\frac{\cos A}{\cos A'} = -\frac{\cos B}{\cos B'} = \frac{\cos C}{\cos C'} = \frac{R'}{R} = k',$$

we should have got the quadratic

$$2\,k'^2 \cos A' \cos B' \cos C' - k' + 1 = 0\,;$$

whose roots are $\dfrac{2\,R'}{R' + SO'}$ and $\dfrac{2\,R'}{R - SO'}$.

The latter is inadmissible as k' should be positive and $R - S'O'$ is negative for an obtuse-angled triangle.

Hence,

$$R = \tfrac{1}{2}\,(R' + SO'.) \qquad \ldots \qquad \ldots \quad (2)$$

We have seen that $R' = \tfrac{1}{2}\,(R + SO)$

Hence

$$SO' = \tfrac{1}{2}\,(3\,R - SO),$$
$$SO = \tfrac{1}{2}\,(3\,R' - SO')\,;$$

and the relations between the two triangles are thus perfectly reciprocal.

If N' be the nine-point centre of $A'B'C'$,

$$SN' = \tfrac{1}{4}\,(3\,R - SO) = \tfrac{1}{2}\,R - \tfrac{1}{4}\,(R + SO)$$
$$= R - \tfrac{1}{2}\,R'.$$

Hence the nine-point circle of $A'B'C'$ and the circum-circle of ABC have internal contact.

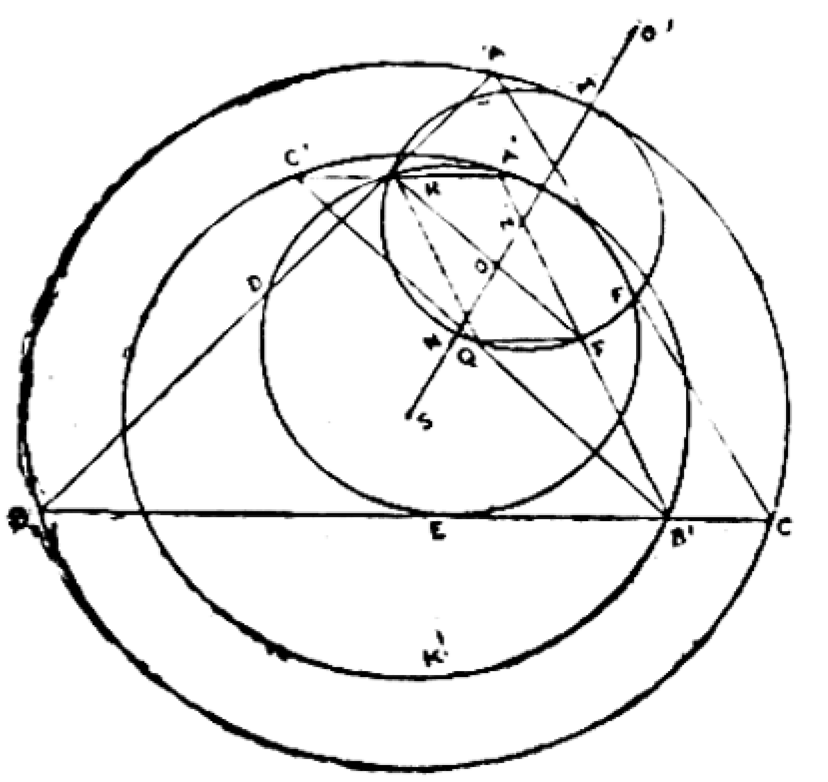
Similarly the reciprocal relation.

[NOTE :—It is easy to see that the second solution

$$k = \frac{2\,R}{R - SO}$$

leads to an obtuse-angled triangle $A''B''C''$, whose vertices are all imaginary. But even in that case, the ortho-centre O'', the nine-point circle, and the circum-circle are all real.

Of the three triangles, Δ, Δ' and Δ'', taking any two, the circum-circle of either touches the nine-point circle of the other, the contact of the circles associated with Δ' and Δ'' being external.]



(2) Let ABC be the acute-angled triangle, O the ortho-centre, and S the circum-centre. Bisect SO at N, then N is the centre of the nine-point circle, and the radius of the nine-point circle is half the radius of the circum-circle of ABC.

Draw the nine-point circle DEF, where D, E, F are the mid.-points of AB, BC, CA respectively. Produce SNO to O' meeting the circle DEF in L, and the circle ABC in M. Then

since all the lines drawn from the ortho-centre of a triangle to the circumference of its circum-circle are bisected by its nine-point circle, we get

$$OL = LM.$$

Make $MO' = ML$. Bisect SO' at N'. With N' as centre and $\frac{1}{2}$ SL as radius describe a circle cutting the three circles drawn with S as centre, and SD, SE, SF as radii at P, Q, R respectively. Also with S as centre and SL as radius draw the circle $A'B'K'C'$. Join PQ, QR, RP. Through Q draw a line $B'QC'$ parallel to PR; through R draw $C'RA'$ parallel to PQ; through P draw $A'PB'$ parallel to QR.

Then $A'B'C'$ is a triangle similar to the triangle PQR; it has the circle $A'B'K'C'$ for its circum-circle, and is such that P, Q, R are the middle points of $A'B'$, $B'C'$, $C'A'$ respectively.

Now in the circle $A'B'K'C'$, S is the centre and P, Q, R are the mid-points of the chords $A'B'$, $B'C'$, $C'A'$. Therefore SP, SQ, SR are perpendicular to $A'B'$, $B'C'$, $C'A'$, respectively.

Hence S is the circum-centre of $A'B'C'$.

The circle PQR is the nine-point circle of the triangle $A'B'C'$. Now since N', the centre of the nine-point circle of the triangle $A'B'C'$, must bisect the line joining S, its circum-centre, to its ortho-centre, and also since $SN' = \frac{1}{2} SO'$, it follows that O' is the ortho-centre of the triangle $A'B'C'$.

Again since the distance of each vertex of a triangle from its ortho-centre is twice the perpendicular distance of the circum-centre from the side opposite to that vertex, we get

$$O'A' = 2SQ,\ O'B' = 2SR,\ O'C' = 2SP.$$

But $SQ = SE = \frac{1}{2} OA$; $SR = SF = \frac{1}{2} OB$; $SP = SD = \frac{1}{2} OC$.

$\therefore \qquad O'A' = OA,\ O'B' = OB,\ O'C' = OC.$

Also since the centres of the two circles $A'B'K'C'$ (the circum-circle of the triangle $A'B'C'$) and DEF (the nine-point circle of the triangle ABC), lie on the straight line SNL, they touch each other at L.

Similarly since the two circles ABC and PQR have their centres on the straight line $SN'M$, they touch each other at M.

Again the radius of the nine-point circle PQR being half of the radius of the circum-circle $A'B'K'C'$ is obviously less than half the radius

of the circle ABC, (*i.e.*, the diameter through M of the circle PQR is less than MS, the radius of the circle ABC). Therefore the circle PQR which touches the circle ABC at M, cuts the three circles drawn with S as centre and SD, SE, SF as radii, at points on the nearer side of S. The triangle A′B′C is therefore situated in a segment smaller than a semi-circle, and hence the angle B′A′C′ which is opposite to the longest side B′C′, is an obtuse angle.

Question 1300.

(M. G. IMAMDAR) :—Prove that the area of a triangle can be got from the formula

$$\Delta = 4\{(\sigma - p)(\sigma - q)\sigma - r)(\sigma - R)\}^{\frac{1}{2}},$$

where $2\sigma = p + q + r + 2R$ and p, q, r, are the perpendiculars from the circum-centre on the sides of the triangle and R is the circum-radius.

Solution (1) *by K. C. Shah,* (2) *by K. Satyanarayana,* (3) *by R. Srinivasan, Ganesan, I. Totadri Iyengar, M. K. Kewalramani, and P. Seetarama Rao.*

(1) Let ρ, r_1, r_2, r_3 be the in-radius and ex-radii of the triangle. If O is the circum-centre, A_1 the mid.-point of BC and TOA_1T_1 the diameter of the circum-circle, then it can be easily proved that

$$2TA_1 = r_2 + r_3.$$

$$\therefore \quad 2p = 2OA_1 = 2TA_1 - 2TO$$

$$= r_2 + r_3 - 2R.$$

Similarly $\quad 2q = r_3 + r_1 - 2R,$

$$2r = r_1 + r_2 - 2R.$$

$$\therefore \quad 4\sigma = 2(r_1 + r_2 + r_3) - 4R.$$

$$\therefore \quad 2\sigma = r_1 + r_2 + r_3 - 2R.$$

Hence $2(\sigma - p) = r_1,\ 2(\sigma - q) = r_2,\ 2(\sigma - r) = r_3;$

and $\quad 2(\sigma - R) = r_1 + r_2 + r_3 - 4R$

$$= \rho, \text{ for } r_1 + r_2 + r_3 - \rho = 4R.$$

$$\therefore \quad 16(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R) = r_1 r_2 r_3 \rho = \Delta^2.$$

$$\therefore \quad \Delta = 4\{(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R)\}^{\frac{1}{2}}.$$

(2) ABC is an acute-angled triangle in which D, E, F are the mid.-points of the sides and S is the circum-centre. Draw DF′ equal and parallel to SF.

Then, since DE, S′F′ are respectively parallel to AB, AC, EDF′, ESF′ are right angles. Hence ESDF′ is cyclic.

Also, from the parallelogram ASF′E,

$$EF' = SA = R$$

and the quadrilateral ESDF′ has its sides equal to p, q, r, R.

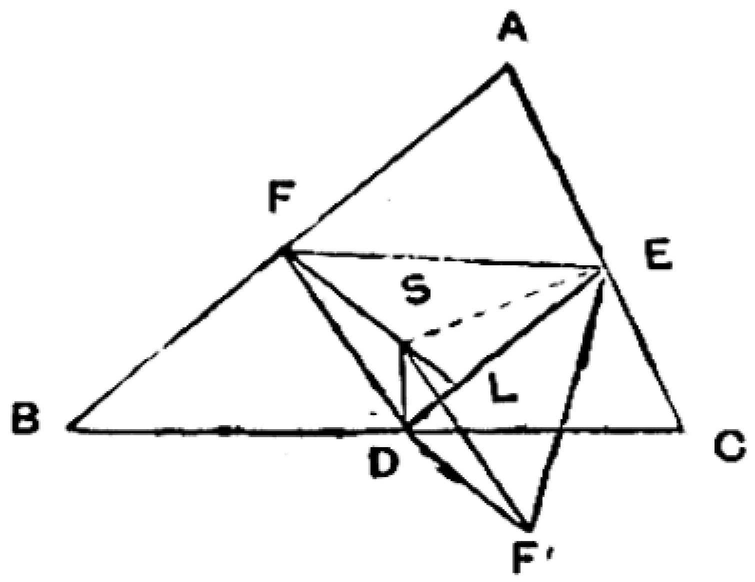
Hence its area $= \sqrt{(\sigma - p)(\sigma - q)(\sigma - r)(\sigma - R)}$.

But if SF cuts DE in L,

$$\Delta\, SFD + \Delta\, SFE = \tfrac{1}{2}\, SF\,(DL + LE) = \Delta\, DEF',$$

$$\therefore \qquad \text{area } ESDF' = \Delta\, DEF = \tfrac{1}{4}\,\Delta\, ABC.$$

Hence the result.



(3) It is evident that

$$p = R\cos A,\ q = R\cos B \text{ and } r = R\cos C.$$

$$\sigma - p = \tfrac{1}{2} R\,(1 + \cos B + \cos C - \cos A)$$

$$= 2R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}.$$

$$\sigma - R = \tfrac{1}{2} R\,(\cos A + \cos B + \cos C - 1)$$

$$= 2R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

$$\therefore \qquad 4\sqrt{\Pi(\sigma - p)} = 4\sqrt{\left\{16R^4.\,\Pi\left(\sin^2\frac{A}{2}\cos^2\frac{A}{2}\right)\right\}}$$

$$= 2R^2 \sin A \sin B \sin C.$$

Questions for Solution.

1367. (S. D. CHOWLA):—Prove that

$$e^{-\beta^2} - e^{-(5\beta)^2} - e^{-(7\beta)^2} + e^{-(11\beta)^2} + e^{-(13\beta)^2}$$

$$- - + + \ldots\ldots$$

$$= \frac{\sqrt{\pi}}{2\sqrt{3}\beta}\left\{ e^{-\left(\frac{\pi}{12\beta}\right)^2} - e^{-\left(\frac{5\pi}{12\beta}\right)^2} - e^{-\left(\frac{7\pi}{12\beta}\right)^2} + + - - \right\}.$$

1368. (R. GOPALASWAMI):—Given four fixed points A, B, C, D, show that there are, in general, two conics through them whose asymptotes include a given angle θ and that the conics are identical for two particular values of θ, one of which is always equal to $\frac{1}{2}\pi$. If these conics be S_1 and S_2, then

(*a*) S_1 and S_2 cut the circle through any three of the four points A, B, C, D at diametrically opposite points.

(*b*) The centres of the conics S_1 and S_2 are the extremities of a diameter of the nine-point conic of ABCD.

1369. (R. GOPALASWAMI):—A_1, B_1, C_1, D_1 are the isogonal conjugates of each of the points A, B, C, D, with respect to the triangles formed by the other three. If for these triangles

$$\{S_1, \rho_1\}\ \{S_2, \rho_2\}\ \{S_3, \rho_3\}\ \{S_4, \rho_4\}$$

are the circum-centres and circum-radii respectively, then

$$\frac{A_1S_1}{\rho_1} = \frac{B_1S_2}{\rho_2} = \frac{C_1S_3}{\rho_3} = \frac{D_1S_4}{\rho_4}.$$

1370. (V. THIRUVENKATACHARYA):—If $\alpha\beta = \pi^2$, show that

(*a*) $$\frac{\pi}{2}\cot\sqrt{\alpha t}\coth\sqrt{\beta t} = \frac{1}{2t} + \frac{\alpha\coth\alpha}{t+\alpha} + \frac{2\alpha\coth 2\alpha}{t+4\alpha} + \ldots\ldots$$

$$+ \frac{\beta\coth\beta}{t-\beta} + \frac{2\beta\coth 2\beta}{t-4\beta} + \ldots\ldots$$

(b) $$\frac{\pi^2}{4}\operatorname{cosec}^2\sqrt{\alpha t}\operatorname{cosech}^2\sqrt{\beta t} = \frac{1}{4t^2} + \frac{\alpha^2\operatorname{cosech}^2\alpha}{(t+\alpha)^2} + \frac{4\alpha^2\operatorname{cosech}^2 2\alpha}{(t+4\alpha)^2} + \ldots\ldots + \frac{\beta^2\operatorname{cosech}^2\beta}{(t-\beta)^2} + \frac{4\beta^2\operatorname{cosech}^2 2\beta}{(t-4\beta)^2} + \ldots\ldots$$

1371. (B. B. BAGI) :—If at the middle points of an acute-angled triangle ABC, perpendiculars to the sides are erected of lengths

$$\tfrac{1}{2}r_1,\ \tfrac{1}{2}r_2,\ \tfrac{1}{2}r_3.$$

all drawn outwards, then the triangle whose vertices are the extremities of these perpendiculars is similar to the triangle of ex-centres.

1372. (K. SATYANARAYANA) :—

(i) If a be any given positive quantity >1, show that as p increases from $-\infty$ to $+\infty$,

$\left[\dfrac{a^p+1}{2}\right]^{\frac{1}{p}}$ increases crossing $\sqrt{a}$ as p crosses 0.

(ii) Find the limit of $\left[\dfrac{a^p+1}{2}\right]^{\frac{1}{p}}$ as

(a) p tends to zero through positive or negative values,

(b) p tends to $\pm\infty$.

1373. (MARTYN THOMAS) :—The equation to a system of curves, in bi-polar co-ordinates, is

$$r^2 - r_1^2 - 2ar + a^2 = 0.$$

Show that the equation to the orthogonal system, in bi-angular co-ordinates, is

$$\tan^2\theta - 2k\tan\theta\operatorname{cosec}2\theta_1 + k^2 = 0.$$

1374. (HANS R. GUPTA) :—If $\dfrac{p_n}{q_n}$ denote the nth convergent of the C. F.

$$\cos\theta - \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -}\ \frac{1}{2\cos\theta -}\ \ldots\ldots,$$

show that $p_n = \cos n\theta$.

Find a similar C. F. for which $p_n = \sin n\theta$.

Index of Notes.

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(*The numbers refer to pages.*)

Index of Questions Solved in Vol. XV.

PRA
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Extract of the Rules.

1. The undermentioned shall be eligible t take books from the Library:—
   - A. Members of the College teaching staff, including the Librarian.
   - B. Members of the establishment of the College.
   - C. Students on the rolls of the College.
   - D. Other persons whether connected with the College or not, who have obtained special permission from the Principal.

2. The maximum number of books that may be borrowed at any time, is

   A ... 10 volumes.

   B & D ... 2 ,,

   C { M.A. ... 6 volumes.
   Hons. ... 4 ,,
   All others ... 2 ,,

3. Books may be retained by A and M.A. and honours students, in class C for one month and all others for fourteen days.

4. Books in any way injured or lost shall be paid for or replaced by the borrower. In case the book belongs to a set or series, unless the book is replaced, the price of the whole set must be paid.

ADDISON, MADRAS.